Multiple Critical Behavior
of Probabilistic Limit Theorems
in the Neighborhood of a Tricritical Point

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Abstract

We derive probabilistic limit theorems that reveal the intricate structure of the phase transitions in a mean-field version of the Blume-Emery-Griffiths model [4]. These probabilistic limit theorems consist of scaling limits for the total spin and moderate deviation principles (MDPs) for the total spin. The model under study is defined by a probability distribution that depends on the parameters n, β, and K, which represent, respectively, the

number of spins, the inverse temperature, and the interaction strength. The intricate structure of the phase transitions is revealed by the existence of 18 scaling limits and 18 MDPs for the total spin. These limit results are obtained as (β, K) converges along appropriate sequences (β_n, K_n) to points belonging to various subsets of the phase diagram, which include a curve of second-order points and a tricritical point. The forms of the limiting densities in the scaling limits and of the rate functions in the MDPs reflect the influence of one or more sets that lie in neighborhoods of the critical points and the tricritical point. Of all the scaling limits, the structure of those near the tricritical point is by far the most complex, exhibiting new types of critical behavior when observed in a limit-theorem phase diagram in the space of the two parameters that parametrize the scaling limits.

1 Introduction

The purpose of this paper is to analyze a new set of phenomena associated with the critical behavior of probabilistic limit theorems for a mean-field version of an important lattice-spin model due to Blume, Emery, and Griffiths [4]. These probabilistic limit theorems consist of scaling limits for the total spin and moderate deviation principles (MDPs) for the total spin.

We will refer to the mean-field model studied in this paper as the BEG model; it is equivalent to the Blume-Emery-Griffiths model on the complete graph on n vertices. In contrast to the mean-field version of the Ising model known as the Curie-Weiss model, whose only phase transition is a continuous, second-order phase transition at the critical inverse temperature [17, §IV.4], the BEG model exhibits both a curve of continuous, second-order points; a curve of discontinuous, first-order points; and a tricritical point, which separates the two curves [22, 29]. It is one of the few models, and certainly one of the simplest, that exhibit this intricate phase-transition structure.

Applications of the Blume-Emery-Griffiths model to a diverse range of physical systems are discussed in [22, §1] and in [29, §3.3], where the model is called the Blume-Emery-Griffiths-Rys model. As the latter reference points out, the model studied in the present paper is actually a mean-field version of a precursor of the Blume-Emery-Griffiths-Rys model due to Blume [3] and Capel [8, 9, 10]. With apologies to these authors, we follow the nomenclature of our earlier paper [22] by referring to this mean-field version as the BEG model.

The BEG model is defined by a probability distribution $P_{n,\beta,K}$, where n equals the number of spins, β is the inverse temperature, and K is the interaction strength. We investigate the complex structure of the phase transitions in the model by deriving 36 different limit results for the total spin S_n as (β, K) converges along appropriate sequences (β_n, K_n) to points belonging to three separate classes: (1) the tricritical point, (2) the curve of second-order points, and (3) the single-phase region lying under that curve. In case 1, we obtain 13 scaling limits and 13 MDPs; in case 2, 4 scaling limits and 4 MDPs; and in case 3, 1 scaling limit and 1 MDP. As we will see, the numbers 13, 4, and 1 represent natural and exhaustive enumerations of three classes of polynomials that arise in the related settings of the scaling limits and the MDPs.

The existence of $18 = 13 + 4 + 1$ scaling limits and 18 MDPs reflects the intricate structure of the phase transitions in the BEG model. It is hoped that our insights can also be applied to other statistical mechanical models that exhibit other types of phase transitions and critical phenomena and thus, presumably, other possibilities for scaling limits of macroscopic random variables like the total spin in the BEG model [19].

Before saying more about the limit theorems in the BEG model and their critical behavior, we summarize a number of facts concerning the phase-transition structure of the model [22]. For $\beta > 0$ and $K > 0$ we denote by $\mathcal{E}_{\beta,K}$ the set of equilibrium macrostates of the model corresponding to the macroscopic variable of the spin per site. In [22] it is proved that there exists a critical inverse temperature $\beta_c = \log 4$ and that for $\beta > 0$ there exists a critical value $K_c(\beta) > 0$ having the following properties.

1. For $\beta > 0$ and $0 < K < K_c(\beta)$, $\mathcal{E}_{\beta,K}$ consists of the unique pure phase 0.

2. For $\beta > 0$ and $K > K_c(\beta)$, $\mathcal{E}_{\beta,K}$ consists of two distinct, nonzero phases.

3. For $0 < \beta \leq \beta_c$, as K increases through $K_c(\beta)$, $\mathcal{E}_{\beta,K}$ undergoes a continuous bifurcation, which corresponds to a second-order phase transition.

4. For $\beta > \beta_c$, as K increases through $K_c(\beta)$, $\mathcal{E}_{\beta,K}$ undergoes a discontinuous bifurcation, which corresponds to a first-order phase transition.

5. The point $(\beta_c, K_c(\beta_c)) = (\log 4, 3/[2 \log 4])$ in the positive quadrant of the β-K plane separates the second-order phase transition noted in item 2 from the first-order phase transition noted in item 4. The point $(\beta_c, K_c(\beta_c))$ is called the tricritical point.

The limit theorems to be considered in the present paper focus on the values of β and K in items 1, 3, and 5. For each such (β, K), $\mathcal{E}_{\beta,K}$ consists of the unique pure phase 0. Figure 1 shows the corresponding portion of the phase diagram, which exhibits three sets A, B, and C. C is the singleton set containing the tricritical point $(\beta_c, K_c(\beta_c))$, B is the curve of second-order points defined by

$$B = \{(\beta, K) \in \mathbb{R}^2 : 0 < \beta < \beta_c, K = K_c(\beta)\}, \tag{1.1}$$

and A is the single-phase region lying under $B \cup C$ and defined by

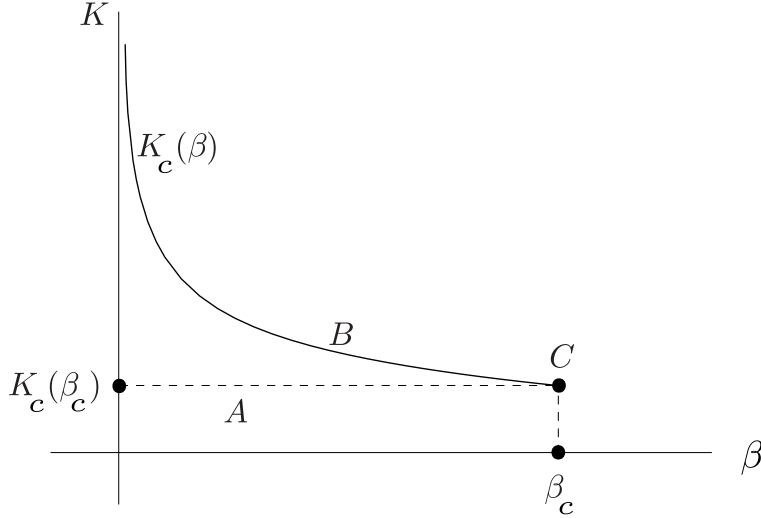$$A = \{(\beta, K) \in \mathbb{R}^2 : 0 < \beta \leq \beta_c, 0 < K < K_c(\beta)\}. \tag{1.2}$$

Figure 1: The sets A, B, and C

In the remainder of this introduction we focus on the scaling limits for the total spin S_n when (β, K) converges to the tricritical point $(\beta_c, K_c(\beta_c))$ along appropriate sequences (β_n, K_n). These scaling limits describe the limiting distribution of $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n} for appropriate choices of $\gamma \in (0, 1/2)$. The simplest sequences for which the full range of scaling limits appear are defined in terms of parameters $\alpha > 0$, $\theta > 0$, $b \neq 0$, and $k \neq 0$ by

$$\beta_n = \log(e^{\beta_c} - b/n^\alpha) \text{ and } K_n = K(\beta_n) - k/n^\theta, \tag{1.3}$$

where $K(\beta) = (e^\beta + 2)/(4\beta)$ for $\beta > 0$. $K(\beta)$ coincides with $K_c(\beta)$ for $0 < \beta \leq \beta_c$ and satisfies $K(\beta) > K_c(\beta)$ for $\beta > \beta_c$ [22, Thms. 3.6, 3.8]. A detailed overview of all the limit theorems in the paper, including those discussed here, is given in the next section.

In each of the scaling limits the form of the limiting density reflects the influence of one or more of the sets A, B, and C that lie in a neighborhood of the tricritical point. The influence of those sets, which depends only on α and θ and not on b or k in (1.3), is shown in Figure 2. In that figure the positive quadrant of the α-θ plane is partitioned into the following sets.

1. Three open sets labeled A, B, and C.

2. Three line segments labeled $A + B$, $A + C$, and $B + C$ that separate the three open sets in item 1.

3. The point equal to $(1/3, 2/3)$ and labeled $A + B + C$ at which the three line segments in item 2 meet.
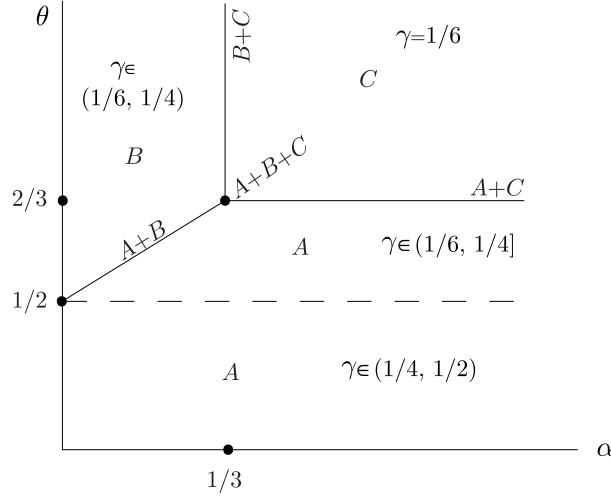

Figure 2: Influence of C, B, and A when $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$

Figure 2 is a limit-theorem phase diagram that summarizes the critical behavior of the scaling limits in a neighborhood of the tricritical point. This critical behavior consists of the following phenomena, which can be verified by examining the statement of the scaling limits in Theorem 7.1.

1. When (α, θ) lies in one of the open sets labeled A, B, or C, then the limiting density in the corresponding scaling limit shows the influence only of that single set. Hence these three open sets correspond to the pure phases of the scaling limits.

2. When (α, θ) lies in one of the line segments labeled $A + B$, $A + C$, or $B + C$, then the limiting density shows the influence of both sets, A and B, A and C, or B and C, respectively. Hence these three line segments correspond to the coexistence of the pure-phase scaling limits noted in item 1.

3. When (α, θ) equals the point labeled $A + B + C$, then the limiting density shows the influence of all three sets A, B, and C. This point is the analogue of the tricritical point in the

standard phase diagram, a portion of which is shown in Figure 1. Indeed, any neighborhood of the tricritical point in the β-K plane contains values of β and K corresponding to all the different phase-transition behaviors of the model. Similarly, any neighborhood of the analogue of the tricritical point in the limit-theorem phase diagram contains values of α and θ corresponding to all the different forms of the scaling limits, which number 13.

4. As (α, θ) crosses any of the line segments labeled $A + B$, $A + C$, or $B + C$, the values of γ in the scaling limits change continuously, which corresponds to a second-order phase transition; by contrast, the forms of the limiting densities change discontinuously, which corresponds to a first-order phase transition.

As noted in items 1, 2, and 3, the influence of the sets upon the forms of the limiting densities reveals a fascinating geometric feature of the BEG model. This feature is completely unexpected because the model has no geometric structure. In fact, each spin interacts equally with all the other spins via a mean-field Hamiltonian, and so the model is independent of dimension. The discussion of the scaling limits given here, including the notion of the influence of a set on the form of the limiting density, will be greatly amplified in the next section.

The scaling limits of $S_n/n^{1-\gamma}$ corresponding to the choices of α and θ in Figure 2 are derived in Theorem 7.1, where we determine the values of α, θ, and γ leading to the various forms of the limit. In Figure 2 the value or range of values of γ are also shown for (α, θ) lying in the sets labeled A, B, and C. The set labeled A is divided into two subsets by the line $\theta = 1/2$; the ranges of values of γ are different in the two subsets.

The three seeds from which the present paper grew are [22], [20], and [14]. In the first paper the phase-transition structure of the BEG model is analyzed. In the second paper scaling limits are proved for a class of models that includes the Curie-Weiss model as a special case. In the third paper 4 different MDPs are obtained for the Curie-Weiss model when the inverse temperature converges to the critical inverse temperature in the model along appropriate sequences β_n. The results derived in the present paper greatly extend both the scaling limits in [20] and the MDPs in [14]. This is the case because the BEG model has a much more intricate structure of phase transitions than the Curie-Weiss model and so exhibits a much richer class both of scaling limits and of MDPs. As we will outline near the end of the next section, both the scaling limits and the MDPs are proved by a unified method.

This unified method is based, in part, on properties of a function $G_{\beta,K}$ defined in (3.4). This function plays a central role in every aspect of the analysis of the BEG model considered in the present paper as well as in its prequel [22]. In summary these are the following.

- The set $\mathcal{E}_{\beta,K}$ of equilibrium macrostates for the BEG model is defined as the set of zeroes of the rate function in the LDP for the $P_{n,\beta,K}$-distributions of S_n/n given in Theorem

3.1. In turn, this set coincides with the set of global minimum points of $G_{\beta,K}$ [see (3.5)]. This characterization of $\mathcal{E}_{\beta,K}$ allowed us to carry out the detailed analysis of the phase-transition structure of the model in [22].

- The canonical free energy $\varphi(\beta, K)$ equals the global minimum value of $G_{\beta,K}$ [see item 2 after (3.4)].

- The distribution of $S_n/n^{1-\gamma}$ can be expressed directly in terms of $G_{\beta,K}$ [Lem. 4.1].

- $G_{\beta,K}$ is the rate function in a second LDP involving S_n/n given in part (b) of Lemma 4.4. The estimates derived from this LDP and given in parts (c) and (d) of the lemma are the key estimates needed to control error terms in the proofs of the scaling limits and the MDPs for $S_n/n^{1-\gamma}$. Lemma 4.4 is the main technical innovation in the paper.

- When a certain quantity w defined in terms of α, θ, and γ equals 0, the 13 different forms of the Taylor expansion of $nG_{\beta_n,K_n}(x/n^\gamma)$ for appropriate sequences (β_n, K_n) and $\gamma \in (0, 1/2)$ yield the 13 different forms of the scaling limits of $S_n/n^{1-\gamma}$ [Thm. 7.1].

- When $w < 0$, the 13 different forms of the Taylor expansion of $n^{1+w}G_{\beta_n,K_n}(x/n^\gamma)$ for appropriate sequences (β_n, K_n) and $\gamma \in (0, 1/2)$ yield the 13 different forms of the MDPs of $S_n/n^{1-\gamma}$ [Thm. 8.3].

This discussion shows that all the magic is in the function $G_{\beta,K}$. The fact that the wide variety of phenomena derived in the present paper and in [22] can be obtained via properties of a single function is an appealing feature of the BEG model. Besides the Curie-Weiss model and generalizations studied in [14, 20, 21, 30] and numerous other papers, this feature is shared with a number of other mean-field models, including a mean-field version of the nearest neighbor Potts model known as the Curie-Weiss-Potts model [25], the mean-field XY Heisenberg model [1], and the Hopfield model of spin glasses and neural networks [31]. These mean-field models have in common the fact that the interaction terms in their Hamiltonians can be written as a quadratic function. Scaling limits and MDPs for these models have either been proved, or in principle could be proved, by techniques similar to those used in the present paper. Some of these techniques are generalized in [11], in which the quadratic term in the Hamiltonian is replaced by the moment generating function of suitable random variables. Other generalizations are given in [5, 6, 23, 24]. The analysis of the equilibrium macrostates and the associated phase transitions in the BEG model, which underlies the present paper, is carried out in [22] using large deviation techniques. While this is an elegant method that provides exact, analytical results, it has the restriction that it works most efficiently in models with long-range interactions, as explained in [2].

The Hopfield model of spin glasses and neural networks has received a great deal of attention, and limit theorems for this model have been actively studied. The Hamiltonian in the Hopfield model can be written as a quadratic function of the overlap parameter, a feature that it shares with the Curie-Weiss model and the BEG model, in which the Hamiltonian can be written as a quadratic function of the spin per site. For the Hopfield model both central limit theorems and non-classical scaling limits for the overlap parameter are studied in [7, 26, 27, 28], and MDPs are studied in [15]. These limit theorems include the cases when the inverse temperature is constant and when the inverse temperature parameter converges to the critical inverse temperature at an appropriate rate [15, 28].

We next preview the contents of the present paper. In section 2 a detailed overview is given of the scaling limits and the MDPs that will be derived. In section 3 we summarize the results in [22] on the structure of the set of equilibrium macrostates of the BEG model and the associated phase transitions. In section 4 we introduce the function $G_{\beta,K}$, properties of which are integral to the proofs of the scaling limits and MDPs. These properties include a formula for the distribution of the total spin in terms of $G_{\beta,K}$ [Lem. 4.1], several forms of the Taylor expansions of $G_{\beta,K}$ that will be used to derive the limit theorems [Thm. 4.3], and two estimates in Lemma 4.4 for controlling error terms in the proofs of the scaling limits and the MDPs.

In sections 5–8 we apply the results in the previous sections to derive the scaling limits and the MDPs. Sections 5 and 6 are devoted to scaling limits for $S_n/n^{1-\gamma}$ when appropriate sequences (β_n, K_n) converge to points $(\beta, K) \in A$ and to points $(\beta, K_c(\beta)) \in B$, where A and B are the sets defined in (1.2) and (1.1). When $(\beta_n, K_n) \to (\beta, K) \in A$ we obtain only 1 scaling limit, which is independent of the sequence (β_n, K_n) [Thm. 5.1]. The situation for $(\beta, K_c(\beta)) \in B$ is much more interesting; for appropriate choices of $(\beta_n, K_n) \to (\beta, K) \in B$, 4 different forms of the scaling limits arise [Thm. 7.1]. The scaling limits proved in these two sections are warm-ups for the even more complicated scaling limits proved in section 7. In that section, for appropriate choices of (β_n, K_n) converging to the tricritical point $(\beta_c, K_c(\beta_c))$ we obtain 13 different forms of the scaling limits [Thm. 7.1]. Finally, in section 8 we obtain 1 MDP for $S_n/n^{1-\gamma}$ when $(\beta_n, K_n) \to (\beta, K) \in A$ [Thm. 8.2], 4 MDPs when $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$ [Thm. 8.1], and 13 MDPs when $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$ [Thm. 8.3]. The MDPs are proved by showing the equivalent Laplace principles, which is carried out by a method closely related to that used to prove the scaling limits in the earlier sections. Being able to prove both classes of limit theorems via a unified method is one of the attractive features of this paper.

2 Overview of the Limit Theorems

This paper is devoted to scaling limits and MDPs for the total spin in the BEG model. In order to highlight the novelty of these results, we introduce some notation. The BEG model is a lattice-spin model defined on the complete graph on n vertices $1, 2, \ldots, n$. The spin at site $j \in \{1, 2, ..., n\}$ is denoted by ω_j, a quantity taking values in $\Lambda = \{-1, 0, 1\}$. The joint distribution of the spins ω_j is defined by a probability measure $P_{n,\beta,K}$ on the configuration space Λ^n [see (3.1)]. The sequence $P_{n,\beta,K}$ for $n \in \mathbb{N}$ defines the canonical ensemble for the BEG model.

Through the particular form of the interactions among the spins, the measures $P_{n,\beta,K}$ incorporate an alignment effect that underlies the phase-transition structure of the model. As $\beta \to 0$, $P_{n,\beta,K}$ converges weakly to the product measure on Λ^n with marginals equal to the uniform measure on Λ. Similarly, as $K \to 0$, $P_{n,\beta,K}$ converges weakly to another product measure on Λ^n. By contrast, as $K \to \infty$, $P_{n,\beta,K}$ concentrates on the configurations ω^+ and ω^- in which the spins are all 1 or -1; by symmetry, as $K \to \infty$, $P_{n,\beta,K}$ converges weakly to the sum of point masses $\frac{1}{2}(\delta_{\omega^+} + \delta_{\omega^-})$. The phase-transition structure of the model reflects the persistence of this alignment effect in the limit $n \to \infty$.

We define $S_n = \sum_{j=1}^n \omega_j$, which represents the total spin. In this paper we will consider numerous weak limits of the distributions of $S_n/n^{1-\gamma}$, where $\gamma \in [0, 1)$. The distributions are with respect to $P_{n,\beta,K}$ for fixed $\beta > 0$ and $K > 0$ and, more generally, with respect to P_{n,β_n,K_n}, where (β_n, K_n) are appropriate sequences converging to specific values of (β, K). The use of P_{n,β_n,K_n} to study weak limits in place of $P_{n,\beta,K}$ is the basic innovation of this paper, which will reveal the intricate phase-transition structure of the model. If ν is a probability measure on $\mathbb{R}$, then the notation $P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow \nu$ means that the distributions of $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n} converge weakly to ν as $n \to \infty$. If f is a nonnegative integrable function on $\mathbb{R}$, then the notation $P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow f\,dx$ means that the distributions of $S_n/n^{1-\gamma}$ converge weakly to the probability measure on $\mathbb{R}$ having a density proportional to f with respect to Lebesgue measure.

The first hint of the intricacy of the phase-transition structure of the BEG model can be seen by examining the law of large numbers and its breakdown, which we consider with respect to $P_{n,\beta,K}$ for fixed $\beta > 0$ and $K > 0$. The intuition is that for sufficiently small $K > 0$ the interactions among the spins are sufficiently weak so that the analogue of the classical law of large numbers holds. However, for sufficiently large $K > 0$ the interactions among the spins are sufficiently strong to cause the classical law of large numbers to break down. This intuition is in fact correct. In [22] it is proved that there exist $K_c(\beta) > 0$, defined for $\beta > 0$, and $z(\beta, K)$,

defined for $\beta > 0$ and $K \geq K_c(\beta)$, in terms of which the following limits hold. The form of the limits for $K = K_c(\beta)$ is given in (2.3) and (2.4).

- For any $\beta > 0$ and $0 < K < K_c(\beta)$

$$P_{n,\beta,K}\{S_n/n \in dx\} \Longrightarrow \delta_0. \tag{2.1}$$

- For any $\beta > 0$ and $K > K_c(\beta)$ we have $z(\beta, K) > 0$ and

$$P_{n,\beta,K}\{S_n/n \in dx\} \Longrightarrow \tfrac{1}{2}\big(\delta_{z(\beta,K)} + \delta_{-z(\beta,K)}\big). \tag{2.2}$$

The proofs of these two limits are indicated at the end of section 3, where they are derived from the LDP given in part (a) of Theorem 3.1.

As we explain in section 3, for each $\beta > 0$ and $K > 0$ the sets of mass points of the limiting measures represent the sets of equilibrium macrostates of the BEG model, which we denote by $\mathcal{E}_{\beta,K}$. Thus, for $\beta > 0$ and $0 < K < K_c(\beta)$, $\mathcal{E}_{\beta,K} = \{0\}$ while for $\beta > 0$ and $K > K_c(\beta)$, $\mathcal{E}_{\beta,K} = \{\pm z(\beta, K)\}$. The quantity $z(\beta, K)$ is a positive, increasing, continuous function for $K > K_c(\beta)$. The limit of $z(\beta, K)$ as $K \to K_c(\beta)^+$ depends on whether $\beta \leq \beta_c$ or $\beta > \beta_c$, where $\beta_c = \log 4$ represents the critical inverse temperature of the model. For $\beta > \beta_c$ we have $z(\beta, K_c(\beta)) > 0$, and

$$\lim_{K \to K_c(\beta)^+} z(\beta, K) = \begin{cases} 0 & \text{if } 0 < \beta \leq \beta_c \\ z(\beta, K_c(\beta)) & \text{if } \beta > \beta_c. \end{cases}$$

Consistent with this limit behavior is the fact that $\mathcal{E}_{\beta,K_c(\beta)}$ equals $\{0\}$ for $0 < \beta \leq \beta_c$ and equals $\{0, \pm z(\beta, K_c(\beta))\}$ for $\beta > \beta_c$. The limit behavior of $z(\beta, K)$ exhibited in the last display shows that the sets $\mathcal{E}_{\beta,K}$ undergo a continuous bifurcation at $K = K_c(\beta)$ for $0 < \beta \leq \beta_c$ and a discontinuous bifurcation at $K = K_c(\beta)$ for $\beta > \beta_c$. From the viewpoint of statistical mechanics, the dual bifurcation behavior of the model corresponds to a continuous, second-order phase transition at $(\beta, K_c(\beta))$ for $0 < \beta \leq \beta_c$ and a discontinuous, first-order phase transition at $(\beta, K_c(\beta))$ for $\beta > \beta_c$. The point $(\beta_c, K_c(\beta_c)) = (\log 4, 3/[2 \log 4])$ separates the second-order phase transition from the first-order phase transition and is called the tricritical point.

The different behavior of the two phase transitions is reflected in the form of the limits of S_n/n when $K = K_c(\beta)$. For $0 < \beta \leq \beta_c$, we have the law of large numbers

$$P_{n,\beta,K_c(\beta)}\{S_n/n \in dx\} \Longrightarrow \delta_0, \tag{2.3}$$

while for $\beta > \beta_c$ the limit is expressed in terms of a measure supported at the three points in $\mathcal{E}_{\beta,K_c(\beta)}$:

$$P_{n,\beta,K_c(\beta)}\{S_n/n \in dx\} \Longrightarrow \lambda_0 \delta_0 + \lambda_1\big(\delta_{z(\beta,K_c(\beta))} + \delta_{-z(\beta,K_c(\beta))}\big). \tag{2.4}$$

In the last limit λ_0 and λ_1 are positive numbers satisfying $\lambda_0 + 2\lambda_1 = 1$ and given explicitly in (4.4). As we point out at the end of section 3, (2.3) follows immediately from the LDP given in part (a) of Theorem 3.1. However, the proof of (2.4) is more subtle and is postponed until after Theorem 4.2.

Further evidence of the intricacy of the phase-transition structure of the model can be seen if one jumps from the context of the law of large numbers and its breakdown to the context of scaling limits for S_n that are related to the central limit theorem and its breakdown. We consider three cases, in all of which $\mathcal{E}_{\beta,K} = \{0\}$. Case 1 is defined by $\beta > 0$ and $0 < K < K_c(\beta)$. For these values of β and K the interactions among the spins are sufficiently weak, and the analogue of the classical central limit theorem holds. As we prove in Theorem 5.1 when $0 < \beta \leq \beta_c$,

$$P_{n,\beta,K}\{S_n/n^{1/2} \in dx\} \Longrightarrow \exp(-c_2 x^2)\, dx, \tag{2.5}$$

where $c_2 = c_2(\beta, K)$ is defined in (5.1). The same limit holds when $\beta > \beta_c$ and $0 < K < K_c(\beta)$.

Case 2 is defined by $0 < \beta < \beta_c$ and $K = K_c(\beta)$. In this case the central limit scaling $n^{1/2}$ in (2.5) must be replaced by $n^{3/4}$, which reflects the onset of long-range order represented by the second-order phase transition at $(\beta, K_c(\beta))$. We have the nonclassical limit

$$P_{n,\beta,K_c(\beta)}\{S_n/n^{3/4} \in dx\} \Longrightarrow \exp(-c_4 x^4)\, dx, \tag{2.6}$$

where $c_4 = c_4(\beta, K) > 0$ is defined in (6.5). The limit in the last display is a special case of one of the limits proved in Theorem 6.1 [see the note after the statement of the theorem].

Case 3 focuses on the tricritical point $(\beta_c, K_c(\beta_c))$. Not only is there an onset of long-range order represented by the second-order phase transition at this point, but also this point separates the second-order phase transition for $\beta < \beta_c$ and the first-order phase transition for $\beta > \beta_c$. This more intricate phase-transition behavior in a neighborhood of the tricritical point is reflected in the replacement of the scaling $n^{3/4}$ for $0 < \beta < \beta_c$ by $n^{5/6}$. In this case

$$P_{n,\beta_c,K_c(\beta_c)}\{S_n/n^{5/6} \in dx\} \Longrightarrow \exp(-c_6 x^6)\, dx, \tag{2.7}$$

where $c_6 = 9/40$. The limit in the last display is a special case of one of the limits proved in Theorem 7.1 [see the note after the statement of the theorem].

For all other values of $\beta > 0$ and $K > 0$ — those satisfying $0 < \beta \leq \beta_c$, $K > K_c(\beta)$ and $\beta > \beta_c$, $K \geq K_c(\beta)$ — the limit theorems have different forms because the set $\mathcal{E}_{\beta,K}$ of equilibrium macrostates consists of more than one point. In both of these cases, for any equilibrium macrostate $\tilde{z}$, $(S_n - n\tilde{z})/n^{1/2}$ satisfies a central-limit-type limit when S_n/n is conditioned to lie in a sufficiently small neighborhood of $\tilde{z}$. The explicit form of the limit is given in part (b) of Theorem 6.6 in [22].

We are now ready to outline the main contribution of this paper, which is to exhibit the intricate probabilistic behavior of the BEG model in neighborhoods of the tricritical point $(\beta_c, K_c(\beta_c))$, second-order points $(\beta, K_c(\beta))$ for $0 < \beta < \beta_c$, and points (β, K) for $0 < \beta \leq \beta_c$ and $0 < K < K_c(\beta)$. We do this by studying scaling limits and MDPs for $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n} for appropriate sequences (β_n, K_n) that converge to points belonging to these three classes and for appropriate choices of $\gamma \in (0, \frac{1}{2}]$. In order to facilitate the discussion, we denote by C the singleton set containing the tricritical point $(\beta_c, K_c(\beta_c))$, by B the curve of second-order points defined by

$$B = \{(\beta, K) \in \mathbb{R}^2 : 0 < \beta < \beta_c, K = K_c(\beta)\},$$

and by A the single-phase region lying under $B \cup C$ and defined by

$$A = \{(\beta, K) \in \mathbb{R}^2 : 0 < \beta \leq \beta_c, 0 < K < K_c(\beta)\}.$$

The sets A, B, and C are shown in Figure 1 in the introduction. In the rest of this section we focus mainly on the scaling limits and MDPs for $S_n/n^{1-\gamma}$ when (β_n, K_n) is an appropriate sequence that converges to $(\beta_c, K_c(\beta_c))$. Scaling limits and MDPs when (β_n, K_n) converges to $(\beta, K_c(\beta)) \in B$ and to $(\beta, K) \in A$ are treated, respectively, in Theorems 6.1 and 8.1 and in Theorems 5.1 and 8.2.

Corresponding to each $(\beta, K) \in A \cup B \cup C$ there exists a unique equilibrium macrostate at 0. We do not consider scaling limits and MDPs in the neighborhoods of other points corresponding to which there exist nonunique equilibrium macrostates. In all or most cases of nonunique equilibrium macrostates, we expect that the scaling limits and MDPs are conditioned limits as in [22, Thm. 6.6(b)] and [14, Thm. 1.1]; however, we have not worked out the details.

Through the limits (2.5), (2.6), and (2.7), each of the sets A, B, and C is associated, respectively, with the term x^2, x^4, and x^6. Specifically, for fixed (β, K)

$$P_{n,\beta,K}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow \begin{cases} \exp(-c_2 x^2)\, dx & \text{with } \gamma = 1/2 \text{ if } (\beta, K) \in A \\ \exp(-c_4 x^4)\, dx & \text{with } \gamma = 1/4 \text{ if } (\beta, K) \in B \\ \exp(-c_6 x^6)\, dx & \text{with } \gamma = 1/6 \text{ if } (\beta, K) \in C, \end{cases} \quad (2.8)$$

where c_2 and c_4 are positive and depend on β and K, and $c_6 = 9/40$. Theorem 7.1 shows that for appropriate sequences (β_n, K_n) converging to $(\beta_c, K_c(\beta_c))$, for appropriate choices of $\gamma \in (0, 1/2)$, and for appropriate coefficients $\tilde{c}_2$, $\tilde{c}_4$, and $\tilde{c}_6$

$$P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow \exp(-\tilde{c}_2 x^2 - \tilde{c}_4 x^4 - \tilde{c}_6 x^6)\, dx. \quad (2.9)$$

As we show in Table 2.1, $G(x) = \tilde{c}_2 x^2 + \tilde{c}_4 x^4 + \tilde{c}_6 x^6$ takes all of the 13 possible forms of an even polynomial of degree 6, 4, or 2 satisfying $G(0) = 0$ and $G(x) \to \infty$ as $|x| \to \infty$. Each of

the 13 cases shows the influence of one or more of the sets C, B, and A through the presence of the term x^6, x^4, or x^2 associated with that set by the limit (2.8). The coefficient $c_6 = 9/40$ is the same as in (2.7), $\bar{c}_4 = 3/16$, and b and k are any nonzero real numbers subject only to the requirement that $\exp(-G)$ is integrable. Because in every case $\gamma \in (0, 1/2)$, the scaling of S_n by $n^{1-\gamma}$ is non-classical.

case	influence	$P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow \exp[-G(x)]\,dx$
1	C	$G(x) = c_6 x^6,\ c_6 > 0$
2	B	$G(x) = b\bar{c}_4 x^4,\ b > 0, \bar{c}_4 > 0$
3	A	$G(x) = k\beta_c x^2,\ k > 0$
4–5	$B + C$	$G(x) = b\bar{c}_4 x^4 + c_6 x^6,\ b \neq 0$
6–7	$A + C$	$G(x) = k\beta_c x^2 + c_6 x^6,\ k \neq 0$
8–9	$A + B$	$G(x) = k\beta_c x^2 + b\bar{c}_4 x^4,\ k \neq 0, b > 0$
10–13	$A + B + C$	$G(x) = k\beta_c x^2 + b\bar{c}_4 x^4 + c_6 x^6, k \neq 0, b \neq 0$

Table 2.1: 13 cases of the scaling limits in (2.9) for (β_n, K_n) in (2.10) and $\gamma \in (0, 1/2)$

The forms of the scaling limits in (2.9) depend crucially on the appropriate choices of the sequences (β_n, K_n) converging to $(\beta_c, K_c(\beta_c))$. The simplest sequences for which all 13 cases of the limit (2.9) arise are defined in terms of parameters $\alpha > 0$, $\theta > 0$, $b \neq 0$, and $k \neq 0$

$$\beta_n = \log(e^{\beta_c} - b/n^\alpha) \text{ and } K_n = K(\beta_n) - k/n^\theta, \tag{2.10}$$

where $K(\beta) = (e^\beta + 2)/(4\beta)$ for $\beta > 0$. $K(\beta)$ coincides with $K_c(\beta)$ for $0 < \beta \leq \beta_c$ and satisfies $K(\beta) > K_c(\beta)$ for $\beta > \beta_c$ [22, Thms. 3.6, 3.8]. Since $\beta_n \to \beta_c$ and since $K(\cdot)$ is continuous, we have $K(\beta_n) \to K_c$; thus the convergence $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$ is valid. In section 7 we will explain how this particular sequence (β_n, K_n) was chosen.

Depending on the signs of b and k, the sequence (β_n, K_n) in (2.10) converges to $(\beta_c, K_c(\beta_c))$ from regions exhibiting markedly different physical behavior. For example, if $b > 0$ and $k > 0$, then $\beta_n < \beta_c$ and $K_n < K(\beta_n)$, and so (β_n, K_n) converges to (β, K) from the region A, corresponding to each point of which there exists a unique equilibrium macrostate [Thm. 3.2(a)]. On the other hand, if $k < 0$, then $K_n > K(\beta_n)$, and so (β_n, K_n) converges to (β, K) from a region of points corresponding to each of which there exist two equilibrium macrostates. If, in addition, $b > 0$, then this region lies above the curve B of second-order points [Thm. 3.2(b)], while if $b < 0$, then this region lies above the curve of first-order points described in Theorem 3.3. Despite the markedly different physical behavior associated with these various regions, all the scaling limits in this paper are proved by a unified method, regardless of the direction of approach of (β_n, K_n) to (β, K). The situation with respect to the MDPs is the same. These remarks concerning the proofs of the scaling limits and the MDPs will be amplified in section 4 after we introduce the tools that will be used in the proofs.

The occurrence of a particular one of the scaling limits enumerated in Table 2.1 depends on γ and on the values of α and θ and thus on the speed at which $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$ and on the direction of approach. Only case 1 expresses the influence of C alone, giving the same limit for (β_n, K_n) in (2.10) as the limit in (2.7), which holds for the constant sequence $(\beta_n, K_n) = (\beta_c, K_c(\beta_c))$. Case 1 occurs if the convergence $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$ is sufficiently fast; namely, $\alpha > 1/3$ and $\theta > 2/3$. Case 2, which expresses the influence of B alone, occurs if the convergence of $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$is sufficiently slow but θ is relatively large compared to α. Case 3, which expresses the influence of A alone, occurs if the convergence is sufficiently slow but, in contrast with case 2, α is relatively large compared to θ. Finally, cases 4–13, which express the influence of more than one set A, B, and C, occur if the convergence of $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$ occurs at an appropriate critical rate. For example, cases 10–13 express the influence of all three sets A, B, and C and so correspond to the most complicated form of the limiting density. This case occurs when $\alpha = 1/3$, $\theta = 2/3$, and $\gamma = 1/6$.

The scaling limits for $S_n/n^{1-\gamma}$ listed in Table 2.1 are analyzed in Theorem 7.1, where we determine the values of α, θ, and γ leading to the 13 different cases. The dependence of (β_n, K_n) in (2.10) upon α and θ is complicated; because β_n is a function of α, K_n is both a function of θ and, through β_n, a function of α. However, as we will see, for the appropriate choice of $\gamma \in (0, 1/2)$, in the expression for the scaling limit of $S_n/n^{1-\gamma}$ the α and the θ decouple in such a way that the limits given in Theorem 7.1 can be read off in a systematic way.

In Figure 2 in the introduction we indicate the subsets of the positive quadrant of the α-θ plane leading to all the cases in Table 2.1. The subsets labeled C, B, and A correspond to cases 1, 2, and 3, respectively, and the subsets labeled $B + C$, $A + C$, $A + B$, and $A + B + C$ correspond to cases 4–5, 6–7, 8–9, and 10–13, respectively. The relationship between the α-θ plane exhibited in Figure 2 and the β-K plane, inside which lies the tricritical point, is that each point in the α-θ plane corresponds, through the formulas for β_n and K_n given in (2.10), to a curve in the β-K plane.

In Figure 3 we exhibit three different curves in the β-K plane, labeled (a), (b), and (abc). These curves correspond to three different choices of α and θ, three different choices of (β_n, K_n) in (2.10), and three different limits in Table 2.1. The curve labeled (a) corresponds to $\alpha = 1$ and $\theta = 1/3$, which in turn corresponds to case 3 of the scaling limit; this case shows the influence only of region A. The curve labeled (b) corresponds to $\alpha = 1/4$ and $\theta = 1$, which in turn corresponds to case 2 of the scaling limit; this case shows the influence only of region B. Finally, the curve labeled (abc) corresponds to $\alpha = 1/3$, $b > 0$, $\theta = 2/3$, and $k > 0$; the associated scaling limit in case 10 shows the influence of all three sets A, B, and C.

It is worth noting a contrast between the scaling limits in (2.8) and those in Table 2.1. In (2.8) the three scaling limits for $S_n/n^{1-\gamma}$ hold with respect to $P_{n,\beta,K}$ for fixed $(\beta, K) \in A$, $(\beta, K) \in B$, and $(\beta, K) \in C$. In each of these three cases the value of γ is fixed to be, respectively, $1/2$, $1/4$, and $1/6$. By contrast, we will see in Theorem 7.1 that in 4 of the 13
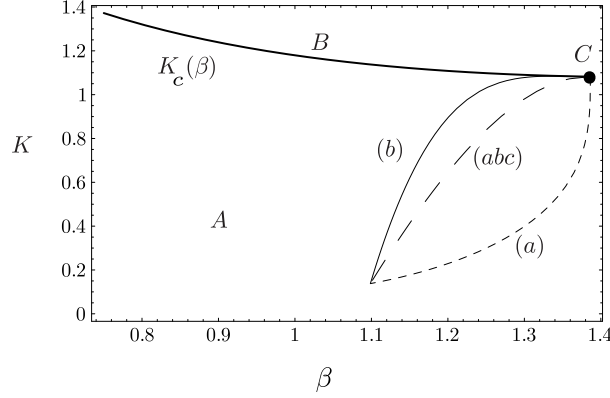
Figure 3: Three choices of (β_n, K_n) that show the influence of A, of B, and of A, B, and C in (2.9)

cases of the scaling limits for $S_n/n^{1-\gamma}$ stated in Table 2.1, the limit theorems hold for a range of values of γ. These are cases 2, 3, 8, and 9. In the other cases, each of which includes the influence of the tricritical point $(\beta_c, K_c(\beta_c))$, γ equals the fixed value $1/6$.

We now make a transition from the scaling limits to the MDPs. As we have seen, the scaling limits state that for appropriate choices of (β_n, K_n) and of $\gamma = \gamma_0$

$$P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma_0} \in dx\} \Longrightarrow \exp[-G(x)]\,dx, \qquad (2.11)$$

where G takes one of the 13 forms in Table 2.1. For any $\gamma \in (0, \gamma_0)$, one can show that if D is any Borel set whose closure does not contain 0, then

$$\lim_{n\to\infty} P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in D\} = 0.$$

A natural question is to determine the rate at which these and related probabilities converge to 0 when (β_n, K_n) is defined in (2.10). In Theorem 8.3 we define a quantity w in terms of α, θ, and γ having the property that when $w < 0$, $S_n/n^{1-\gamma}$ satisfies an MDP with exponential speed n^{-w} and rate function $G(x) - \bar{G}$, where G is the same function appearing in (2.11) and $\bar{G} = \inf_{y \in \mathbb{R}} G(y)$. This MDP implies that for suitable sets D

$$P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in D\} \to 0 \text{ like } \exp[-n^{-w}\inf_{x\in D}(G(x) - \bar{G})].$$

In order to emphasize the similarity with the scaling limits, we summarize this class of MDPs by the formal notation

$$P_{n,\beta_n,K_n}\{S_n n^{1-\gamma} \in dx\} \asymp \exp[-n^{-w}G(x)], \qquad (2.12)$$

in which the constant $\bar{G}$ is not shown.

The situation with the MDPs is completely analogous to the situation for the scaling limits. Specifically, as we exhibit in Table 2.2, there are 13 cases of the MDP (2.12), each of which shows the influence of one or more of the sets C, B, and A depending on the speed at which the sequence (β_n, K_n) defined in (2.10) converges to $(\beta_c, K_c(\beta_c))$ and on its direction of approach. The coefficient $c_6 = 9/40$ is the same as in (2.7), $\bar{c}_4 = 3/16$, and b and k are the nonzero real numbers appearing in (2.10) and subject only to the requirement that $G(x) \to \infty$ as $|x| \to \infty$. The MDPs for $S_n/n^{1-\gamma}$ listed in Table 2.2 are analyzed in Theorem 8.3, where we determine the values of α, θ, and γ that lead to each of the cases.

case	influence	$P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\} \asymp \exp[-n^{-w}G(x)]\,dx$
1	C	$G(x) = c_6 x^6, \ c_6 > 0$
2	B	$G(x) = b\bar{c}_4 x^4, \ b > 0, \bar{c}_4 > 0$
3	A	$G(x) = k\beta_c x^2, \ k > 0$
4–5	$B + C$	$G(x) = b\bar{c}_4 x^4 + c_6 x^6, \ b \neq 0$
6–7	$A + C$	$G(x) = k\beta_c x^2 + c_6 x^6, \ k \neq 0$
8–9	$A + B$	$G(x) = k\beta_c x^2 + b\bar{c}_4 x^4, \ k \neq 0, b > 0$
10–13	$A + B + C$	$G(x) = k\beta_c x_2 + b\bar{c}_4 x^4 + c_6 x^6, k \neq 0, b \neq 0$

Table 2.2: 13 cases of the MDPs in (2.12) for (β_n, K_n) in (2.10) and $\gamma \in (0, 1/2)$

The MDPs for $S_n/n^{1-\gamma}$ have an unexpected consequence concerning a new class of distribution limits for $S_n/n^{1-\gamma}$ that give deeper insight into the fine structure of the phase transitions in a neighborhood of the tricritical point. In an effort to understand the physical significance of these new limits, analogs of them are now being investigated for a class of non-mean-field models, including the Blume-Emery-Griffiths model [19]. In order to appreciate these new results, we first consider a consequence of the large deviation principle stated in part (a) of Theorem 3.1. Since $\mathcal{E}_{\beta,K} = \{0\}$ for $(\beta, K) \in A \cup B \cup C$, it follows that for any positive sequence $(\beta_n, K_n) \to (\beta, K) \in A \cup B \cup C$

$$P_{n,\beta_n,K_n}\{S_n/n \in dx\} \Longrightarrow \delta_0.$$

The MDPs for $S_n/n^{1-\gamma}$ listed in Table 2.2 lead to refinements of this limit for $(\beta_c, K_c(\beta_c)) \in C$ in those cases in which the set of global minimum points of G contains nonzero points. These are precisely the cases in which the coefficients of G are not all positive: cases 5 ($b < 0$), 7 ($k < 0$), 9 ($k < 0$), 11 ($k < 0, b > 0$), 12 ($k > 0, b < 0$), and 13 ($k < 0, b < 0$). In all these cases except for case 12, the set of global minimum points of G consists of two symmetric, nonzero points $\pm x(b, k)$. Hence, using the appropriate value of γ and the appropriate sequence (β_n, K_n) given in Theorem 8.3, we deduce from the corresponding MDP the limit

$$P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow \tfrac{1}{2}\big(\delta_{x(b,k)} + \delta_{-x(b,k)}\big). \tag{2.13}$$

In each of these cases (β_n, K_n) approaches $(\beta_c, K_c(\beta_c))$ from a region of points (β, K) corresponding to each of which there exist two equilibrium macrostates $\pm z(\beta, k)$ [Thms. 3.2(b), 3.3(c)]. As we have already seen, for each (β, K) in this region the limit (2.2) holds. The new limit (2.13) shows that as $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$ from this two-phase region, the model retains a trace of the two equilibrium macrostates $\pm z(\beta, K)$, replacing them by the quantities $\pm x(b, k)$. The physical significance of this limit as well as the limit (2.14) to be stated in the next paragraph is currently under investigation [19]. A similar phenomenon occurs in case 4 of Theorem 8.1, which proves MDPs for $S_n/n^{1-\gamma}$ for appropriate sequences (β_n, K_n) converging to (β, K) lying in the curve B of second-order points.

The situation in case 12 in Table 2.2 ($k > 0$, $b < 0$) is even more fascinating than in the other cases. For fixed $b < 0$, fixed $n \in \mathbb{N}$, and decreasing $k > 0$, the set of global minimum points of G undergoes a discontinuous bifurcation, changing from a unique global minimum point at 0 for k large, to three global minimum points at $0, \pm x(b, k)$ for a critical value of $k = \text{const} \cdot b^2$, to two global minimum points at $\pm x(b, k)$ for k small. As k decreases, (β_n, K_n) crosses the first-order critical curve from below; the changing forms of the sets of global minimum points of G replicate the changing forms of $\mathcal{E}_{\beta, K}$ for fixed $\beta > \beta_c$ and increasing $K > 0$ [Thm. 3.3]. In particular, when the set of global minimum points of G equals $\{0, \pm x(b, k)\}$, the MDP corresponding to case 12 together with other information yields the limit

$$P_{n, \beta_n, K_n}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow \bar{\lambda}_0 \delta_0 + \bar{\lambda}_1 \big(\delta_{x(b,k)} + \delta_{-x(b,k)}\big), \tag{2.14}$$

where $\bar{\lambda}_0$ and $\bar{\lambda}_1$ are positive numbers satisfying $\bar{\lambda}_0 + 2\bar{\lambda}_1 = 1$. This limit is reminiscent of the limit (2.4), in which the equilibrium macrostates $\pm z(\beta, K)$ are replaced by their traces $\pm x(b, k)$.

Although in general the values of α, θ, and γ leading to each of the 13 cases of the MDPs in Table 2.2 differ from the values of these parameters leading to the corresponding case of the scaling limit in Table 2.1, the tables have a number of obvious similarities. This resemblance between the two tables reaches deeper. In fact, both sets of results are proved by a unified method. In order to explain this, let f be any bounded, continuous function mapping $\mathbb{R}$ into $\mathbb{R}$ and let (β_n, K_n) be any positive sequence. The starting point of the proofs of both the scaling limits and the MDPs [see Lem. 4.1] is that whenever $\gamma \in (0, 1/2)$, we have

$$E\{f(S_n/n^{1-\gamma} + \varepsilon_n)\} = \frac{1}{Z_n} \cdot \int_{\mathbb{R}} f(x) \, \exp[-n G_{\beta_n, K_n}(x/n^\gamma)] \, dx. \tag{2.15}$$

The function $G_{\beta, K}$ in this display is defined in (3.4); its global minimum value equals the canonical free energy for the model. In addition, ε_n represents a sequence of random variables that converges to 0 as $n \to \infty$, and Z_n is a normalizing constant.

The quantity w in the MDP (2.12) is defined by $w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}$. This quantity also plays a key role in the scaling limits for $S_n/n^{1-\gamma}$, which like the MDPs arise

from the choice of (β_n, K_n) in (2.10). When $w = 0$, the scaling limits listed in Table 2.1 follow at least formally from (2.15) and the fact that for each $x \in \mathbb{R}$

$$\lim_{n \to \infty} n G_{\beta_n, K_n}(x/n^\gamma) = G(x).$$

The proof of this limit relies on an analysis of the Taylor expansion of G_{β_n, K_n} at 0, which has 13 different forms depending on the choices of γ and of the parameters α and θ appearing in the definition (2.10) of (β_n, K_n). Details are given in Theorem 7.1.

We now assume that $w < 0$. Given ψ be any bounded, continuous function, we substitute $f = \exp(n^{-w}\psi)$ into (2.15), obtaining

$$E\{\exp[n^{-w}\psi(S_n/n^{1-\gamma} + \varepsilon_n)]\}$$
$$= \frac{1}{Z_n} \cdot \int_{\mathbb{R}} \exp\left[n^{-w}\left\{\psi(x) - n^{1+w}G_{\beta_n, K_n}(x/n^\gamma)\right\}\right] dx.$$

When $w < 0$, the last display, the fact that for each $x \in \mathbb{R}$

$$\lim_{n \to \infty} n^{1+w} G_{\beta_n, K_n}(x/n^\gamma) = G(x),$$

and the fact that $\varepsilon_n \to 0$ in probability at a rate faster than $\exp(-n^{-w})$ give the formal asymptotics

$$E\{\exp[n^{-w}\psi(S_n/n^{1-\gamma} + \varepsilon_n)]\}$$
$$\approx \int_{\mathbb{R}} \exp\left[n^{-w}\left\{\psi(x) - (G(x) - \bar{G})\right\}\right] dx$$
$$\approx \exp\left[n^{-w} \sup_{x \in \mathbb{R}} \left\{\psi(x) - (G(x) - \bar{G})\right\}\right],$$

where $\bar{G} = \inf_{y \in \mathbb{R}} G(y)$. In section 8 we show to convert this formal calculation into a limit known as the Laplace principle, which is equivalent to the MDPs for $S_n/n^{1-\gamma}$ listed in Table 2.2. As in the proof of the scaling limits, the proof of the Laplace limit relies on an analysis of the Taylor expansion of G_{β_n, K_n} at 0. Despite the similarity in the proofs of the scaling limits and the Laplace principles, the proof of the latter is much more delicate, requiring additional estimates not needed in the proof of the former.

We start our analysis of the BEG model in the next section.

3 Phase-Transition Structure of the BEG Model

After defining the BEG model, we summarize its phase-transition structure in Theorems 3.2 and 3.3. In (3.4) we introduce the function $G_{\beta, K}$, in terms of which the scaling limits and the MDPs for $S_n/n^{1-\gamma}$ will be deduced later in the paper.

The BEG model is a lattice-spin model defined on the complete graph on n vertices $1, 2, \ldots, n$. The spin at site $j \in \{1, 2, \ldots, n\}$ is denoted by ω_j, a quantity taking values in $\Lambda = \{-1, 0, 1\}$. The configuration space for the model is the set Λ^n containing all sequences $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ with each $\omega_j \in \Lambda$. In terms of a positive parameter K representing the interaction strength, the Hamiltonian is defined by

$$H_{n,K}(\omega) = \sum_{j=1}^{n} \omega_j^2 - \frac{K}{n} \left(\sum_{j=1}^{n} \omega_j \right)^2$$

for each $\omega \in \Lambda^n$. For $n \in \mathbb{N}$, inverse temperature $\beta > 0$, and $K > 0$, the canonical ensemble for the BEG model is the sequence of probability measures that assign to each subset B of Λ^n the probability

$$P_{n,\beta,K}(B) = \frac{1}{Z_n(\beta, K)} \cdot \int_B \exp[-\beta H_{n,K}] \, dP_n. \tag{3.1}$$

In this formula P_n is the product measure on Λ^n with identical one-dimensional marginals

$$\rho = \tfrac{1}{3}(\delta_{-1} + \delta_0 + \delta_1),$$

and $Z_n(\beta, K)$ is the normalizing constant $\int_{\Lambda^n} \exp[-\beta H_{n,K}] dP_n$.

In [22] the analysis of the canonical ensemble $P_{n,\beta,K}$ was facilitated by expressing it in the form of a Curie-Weiss-type model. This is done by absorbing the noninteracting component of the Hamiltonian into the product measure P_n, obtaining

$$P_{n,\beta,K}(d\omega) = \frac{1}{\tilde{Z}_n(\beta, K)} \cdot \exp\left[n\beta K \left(\frac{S_n(\omega)}{n} \right)^2 \right] P_{n,\beta}(d\omega). \tag{3.2}$$

In this formula $S_n(\omega)$ equals the total spin $\sum_{j=1}^{n} \omega_j$, $P_{n,\beta}$ is the product measure on Λ^n with identical one-dimensional marginals

$$\rho_\beta(d\omega_j) = \frac{1}{Z(\beta)} \cdot \exp(-\beta \omega_j^2) \, \rho(d\omega_j), \tag{3.3}$$

$Z(\beta)$ is the normalizing constant $\int_\Lambda \exp(-\beta \omega_j^2)\rho(d\omega_j) = 1 + 2e^{-\beta}$, and $\tilde{Z}_n(\beta, K)$ is the normalizing constant $[Z(\beta)]^n/Z_n(\beta, K)$.

Although $P_{n,\beta,K}$ has the form of a Curie-Weiss model when rewritten as in (3.2), it is much more complicated because of the β-dependent product measure $P_{n,\beta}$ and the presence of the parameter K. These complications introduce new features not present in the Curie-Weiss model [17, §IV.4, §V.9]; these features include the existence of a second-order phase transition for all

sufficiently small $\beta > 0$ and all sufficiently large $K > 0$ and a first-order phase transition for all sufficiently large $\beta > 0$ and all sufficiently large $K > 0$. The existence of a second-order phase transition and a first-order phase transition also implies the existence of a tricritical point, which separates the two phase transitions and is one of the main focuses of the present paper.

The starting point of the analysis of the phase-transition structure of the BEG model is the large deviation principle (LDP) satisfied by the spin per site S_n/n with respect to $P_{n,\beta,K}$. In order to state the form of the rate function, we introduce the cumulant generating function c_β of the measure ρ_β defined in (3.3); for $t \in \mathbb{R}$ this function is defined by

$$
\begin{aligned}
c_\beta(t) &= \log \int_\Lambda \exp(t\omega_1)\, \rho_\beta(d\omega_1) \\
&= \log\left[\frac{1 + e^{-\beta}(e^t + e^{-t})}{1 + 2e^{-\beta}}\right].
\end{aligned}
$$

We also introduce the Legendre-Fenchel transform of c_β, which is defined for $z \in [-1, 1]$ by

$$
J_\beta(z) = \sup_{t \in \mathbb{R}}\{tz - c_\beta(t)\};
$$

$J_\beta(z)$ is finite for $z \in [-1, 1]$. J_β is the rate function in Cramér's theorem, which is the LDP for S_n/n with respect to the product measures $P_{n,\beta}$ [17, Thm. II.4.1] and is one of the components of the proof of the LDP for S_n/n with respect to $P_{n,\beta,K}$. This LDP and a related limit are stated in parts (a) and (b) of the next theorem. Part (a) is proved in Theorem 3.3 in [22], and part (b) in Theorem 2.4 in [18].

Theorem 3.1. *For all $\beta > 0$ and $K > 0$ the following conclusions hold.*

(a) *With respect to the canonical ensemble $P_{n,\beta,K}$, S_n/n satisfies the LDP on $[-1, 1]$ with exponential speed n and rate function*

$$
I_{\beta,K}(z) = J_\beta(z) - \beta K z^2 - \inf_{y \in \mathbb{R}}\{J_\beta(y) - \beta K y^2\}.
$$

(b) *We define the canonical free energy*

$$
\varphi(\beta, K) = -\lim_{n \to \infty} \frac{1}{n} \log Z_n(\beta, K),
$$

where $Z_n(\beta, K)$ is the normalizing constant in (3.1). Then $\varphi(\beta, K) = \inf_{y \in \mathbb{R}}\{J_\beta(y) - \beta K y^2\}$.

The LDP in part (a) of the theorem implies that those $z \in [-1, 1]$ satisfying $I_{\beta,K}(z) > 0$ have an exponentially small probability of being observed in the canonical ensemble. Hence we define the set of equilibrium macrostates by

$$
\mathcal{E}_{\beta,K} = \{z \in [-1, 1] : I_{\beta,K}(z) = 0\}.
$$

In [22] we used the notation $\tilde{\mathcal{E}}_{\beta,K}$ to describe this set, using the notation $\mathcal{E}_{\beta,K}$ to describe a different but related set of equilibrium macrostates. In the present paper we write $\mathcal{E}_{\beta,K}$ instead of $\tilde{\mathcal{E}}_{\beta,K}$ in order to simplify the notation.

For $z \in \mathbb{R}$ we define

$$G_{\beta,K}(z) = \beta K z^2 - c_\beta(2\beta K z). \tag{3.4}$$

The calculation of the zeroes of $I_{\beta,K}$ — equivalently, the global minimum points of $J_{\beta,K}(z) - \beta K z^2$ — is greatly facilitated by the following observations made in Proposition 3.4 in [22]:

1. The global minimum points of $J_{\beta,K}(z) - \beta K z^2$ coincide with the global minimum points of $G_{\beta,K}$, which are much easier to calculate.

2. The minimum values $\min_{z \in \mathbb{R}}\{J_{\beta,K}(z) - \beta K z^2\}$ and $\min_{z \in \mathbb{R}} G_{\beta,K}(z)$ coincide and both equal the canonical free energy $\varphi(\beta, K)$ defined in part (b) of Theorem 3.1.

Item 1 gives the alternate characterization that

$$\mathcal{E}_{\beta,K} = \{z \in [-1, 1] : z \text{ minimizes } G_{\beta,K}(z)\}. \tag{3.5}$$

In the context of Curie-Weiss-type models, the form of $G_{\beta,K}$ is explained on page 2247 of [22].

As shown in the next two theorems, the structure of $\mathcal{E}_{\beta,K}$ depends on the relationship between β and the critical value $\beta_c = \log 4$. We first describe $\mathcal{E}_{\beta,K}$ for $0 < \beta \leq \beta_c$ and then for $\beta > \beta_c$. In the first case $\mathcal{E}_{\beta,K}$ undergoes a continuous bifurcation as K increases through the critical value $K_c(\beta)$ defined in (3.6); physically, this bifurcation corresponds to a second-order phase transition. The following theorem is proved in Theorem 3.6 in [22].

Theorem 3.2. *For $0 < \beta \leq \beta_c$, we define*

$$K_c(\beta) = \frac{1}{2\beta c_\beta''(0)} = \frac{e^\beta + 2}{4\beta}. \tag{3.6}$$

For these values of β, $\mathcal{E}_{\beta,K}$ has the following structure.

(a) For $0 < K \leq K_c(\beta)$, $\mathcal{E}_{\beta,K} = \{0\}$.

(b) For $K > K_c(\beta)$, there exists $z(\beta, K) > 0$ such that $\mathcal{E}_{\beta,K} = \{\pm z(\beta, K)\}$.

(c) $z(\beta, K)$ is a positive, increasing, continuous function for $K > K_c(\beta)$, and as $K \to (K_c(\beta))^+$, $z(\beta, K) \to 0$. Therefore, $\mathcal{E}_{\beta,K}$ exhibits a continuous bifurcation at $K_c(\beta)$.

For $\beta \in (0, \beta_c)$, the curve $(\beta, K_c(\beta))$ is the curve of second-order points. As we will see in a moment, for $\beta \in (\beta_c, \infty)$ the BEG model also has a curve of first-order points, which we denote by the same notation $(\beta, K_c(\beta))$. In order to simplify the notation, we do not follow the convention in [22], where we distinguished between the second-order phase transition and the

first-order phase transition by writing $K_c(\beta)$ for $0 < \beta \le \beta_c$ as $K_c^{(2)}(\beta)$ and writing $K_c(\beta)$ for $\beta > \beta_c$ as $K_c^{(1)}(\beta)$.

We now describe $\mathcal{E}_{\beta,K}$ for $\beta > \beta_c$. In this case $\mathcal{E}_{\beta,K}$ undergoes a discontinuous bifurcation as K increases through an implicitly defined critical value. Physically, this bifurcation corresponds to a first-order phase transition. The following theorem is proved in Theorem 3.8 in [22].

Theorem 3.3. *For all $\beta > \beta_c$, $\mathcal{E}_{\beta,K}$ has the following structure in terms of the quantity $K_c(\beta)$, denoted by $K_c^{(1)}(\beta)$ in [22] and defined implicitly for $\beta > \beta_c$ on page 2231 of [22].*
  (a) *For $0 < K < K_c(\beta)$, $\mathcal{E}_{\beta,K} = \{0\}$.*
  (b) *There exists $z(\beta, K_c(\beta)) > 0$ such that $\mathcal{E}_{\beta,K_c(\beta)} = \{0, \pm z(\beta, K_c(\beta))\}$.*
  (c) *For $K > K_c(\beta)$ there exists $z(\beta, K) > 0$ such that $\mathcal{E}_{\beta,K} = \{\pm z(\beta, K)\}$.*
  (d) *$z(\beta, K)$ is a positive, increasing, continuous function for $K \ge K_c(\beta)$, and as $K \to K_c(\beta)^+$, $z(\beta, K) \to z(\beta, K_c(\beta)) > 0$. Therefore, $\mathcal{E}_{\beta,K}$ exhibits a discontinuous bifurcation at $K_c(\beta)$.*

We end this section by outlining the proofs of the laws of large numbers in (2.1) and (2.3) and its breakdown in (2.2). The upper large deviation bound in the LDP stated in part (a) of Theorem 3.1 implies that for any $\beta > 0$ and $K > 0$ the limiting mass of S_n/n with respect to $P_{n,\beta,K}$ concentrates on the elements of $\mathcal{E}_{\beta,K}$. According to Theorems 3.2(a) and 3.3(a), $\mathcal{E}_{\beta,K} = \{0\}$ when $0 < \beta \le \beta_c$ and $0 < K \le K_c(\beta)$ and when $\beta > \beta_c$ and $K < K_c(\beta)$. For these values of β and K, the laws of large numbers in (2.1) and (2.3) follow immediately. For $\beta > 0$ and $K > K_c(\beta)$, we have $\mathcal{E}_{\beta,K} = \{\pm z(\beta, K)\}$, and so by symmetry the limit (2.2) follows. The proof of the limit (2.4) is postponed until after Theorem 4.2 because it requires more detailed information about the elements of $\mathcal{E}_{\beta,K}$ when $\beta > \beta_c$ and $K = K_c(\beta)$.

In the next section we present additional properties of the function $G_{\beta,K}$ introduced in (3.4). These properties will be used in later sections to prove the scaling limits and the MDPs for $S_n/n^{1-\gamma}$.

4 Properties of $G_{\beta,K}$

As we saw in (3.5), the global minimum points of

$$
\begin{aligned}
G_{\beta,K}(z) &= \beta K z^2 - c_\beta(2\beta K z) \\
&= \beta K z^2 - \log\left[\frac{1 + e^{-\beta}(e^{2\beta K z} + e^{-2\beta K z})}{1 + 2e^{-\beta}}\right]
\end{aligned}
$$

coincide with the elements of $\mathcal{E}_{\beta,K}$, the set of equilibrium macrostates for the BEG model. In this section we study further properties of $G_{\beta,K}$ that will be used in later sections to prove the

scaling limits and the MDPs for $S_n/n^{1-\gamma}$ with respect to $P_{n,\beta,K}$ and with respect to P_{n,β_n,K_n} for appropriate sequence (β_n, K_n) and for appropriate choices of γ.

We first show that for any $\gamma \in [0, 1)$ the P_{n,β_n,K_n}-distribution of $S_n/n^{1-\gamma}$ can be expressed in terms of G_{β_n,K_n} and an independent normal random variable. The next lemma can be proved like Lemma 3.3 in [20], which applies to the Curie-Weiss model, or like Lemma 3.2 in [25], which applies to the Curie-Weiss-Potts model. In an equivalent form, the next lemma is well known in the literature as the Hubbard-Stratonovich transformation, where it is invoked to analyze models with quadratic Hamiltonians (see, e.g., [1, p. 2363]). After the statement of the lemma, we outline how we will use it in order to deduce the scaling limits of $S_n/n^{1-\gamma}$.

Lemma 4.1. *Given a positive sequence* (β_n, K_n), *let* W_n *be a sequence of* $N(0, (2\beta_n K_n)^{-1})$ *random variables defined on a probability space* $(\Omega, \mathcal{F}, Q)$. *Then for any* $\gamma \in [0, 1)$ *and any bounded, measurable function* f

$$\int_{\Lambda^n \times \Omega} f\left(\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}}\right) d(P_{n,\beta_n,K_n} \times Q) \tag{4.1}$$

$$= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx} \cdot \int_{\mathbb{R}} f(x)\, \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx.$$

As we will see in Theorems 5.1, 6.1, and 7.1, the scaling limits have different forms depending on which of the following three sets (β, K) lies in: the singleton set C containing the tricritical point $(\beta_c, K_c(\beta_c))$, the curve B of second-order points

$$B = \{(\beta, K) \in \mathbb{R}^2 : 0 < \beta < \beta_c, K = K_c(\beta)\},$$

and the single-phase region

$$A = \{(\beta, K) \in \mathbb{R}^2 : 0 < \beta \le \beta_c, 0 < K < K_c(\beta)\}.$$

These sets are shown in Figure 1 in the introduction.

We now indicate how we will use Lemma 4.1 to prove the scaling limits of $S_n/n^{1-\gamma}$ for $\gamma \in (0, 1/2]$. Let (β_n, K_n) be a suitable positive sequence converging to $(\beta, K) \in A \cup B \cup C$. Assume that (β_n, K_n) and γ are chosen so that the limit of the right hand side of (4.1) exists as $n \to \infty$. We first consider $\gamma < 1/2$. Since β_n and K_n are bounded and uniformly positive over n, rewriting the limit of the left hand side in terms of characteristic functions shows that $W_n/n^{1/2-\gamma}$ does not contribute. Hence it follows that

$$\lim_{n\to\infty} \int_{\Lambda^n} f(S_n/n^{1-\gamma})\, dP_{n,\beta_n,K_n} \tag{4.2}$$

$$= \lim_{n\to\infty} \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx} \cdot \int_{\mathbb{R}} f(x)\, \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx.$$

From this formula we will be able to determine the scaling limits of $S_n/n^{1-\gamma}$ when $(\beta_n, K_n) \rightarrow (\beta, K) \in B \cup C$ [Thms. 6.1, 7.1]. Using an analogous formula, we will be able to determine the MDPs of $S_n/n^{1-\gamma}$ when $(\beta_n, K_n) \rightarrow (\beta, K) \in B \cup C$ [Thms. 8.1, 8.3].

Now consider $\gamma = 1/2$, which corresponds to the central-limit-type scaling for S_n in (2.5). In this case (4.1) yields

$$\lim_{n \to \infty} \int_{\Lambda^n \times \Omega} f(S_n/n^{1/2} + W_n) \, d(P_{n,\beta_n,K_n} \times Q) \tag{4.3}$$
$$= \lim_{n \to \infty} \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^{1/2})] \, dx} \cdot \int_{\mathbb{R}} f(x) \, \exp[-nG_{\beta_n,K_n}(x/n^{1/2})] \, dx.$$

In contrast to when $\gamma \in (0, 1/2)$, W_n now contributes to the limit. Again the use of characteristic functions enables one to determine the scaling limit of $S_n/n^{1/2}$ when $(\beta_n, K_n) \rightarrow (\beta, K) \in A$ [Thm. 5.1].

Formulas (4.2) and (4.3) suggest how to proceed in proving the scaling limits of $S_n/n^{1-\gamma}$. First consider (β_n, K_n) for which G_{β_n,K_n} has a unique global minimum point at 0 [Thms. 3.2(a), 3.3(a)]. As (4.2) and (4.3) suggest, the forms of the scaling limits of $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n} depend on the forms of the Taylor expansions of G_{β_n,K_n} in the neighborhood of the global minimum point 0. One of the attractive features of our analysis is that the same Taylor expansions can be used to handle sequences (β_n, K_n) for which G_{β_n,K_n} has nonunique global minimum points. Such sequences arise naturally in the scaling limits and the MDPs to be proved later in the paper; in fact, it is precisely such sequences for which the MDPs yield the new class of distribution limits of the form (2.13) and (2.14). What makes it possible to use the same Taylor expansions regardless of the nature of the global minimum points of G_{β_n,K_n} is Lemma 4.4, the main technical innovation in this paper.

Preliminary information on the forms of the relevant Taylor expansions is presented in Theorems 4.2 and 4.3. In the proofs of the scaling limits, in order to justify replacing $nG_{\beta_n,K_n}(x/n^{\gamma})$ in (4.2) by n times the Taylor expansion evaluated at x/n^{γ} and taking limits under the integral, one invokes the dominated convergence theorem, for which the appropriate bounding function depends on the particular sequence (β_n, K_n). This will be handled on a case-by-case basis in subsequent sections. Finally, one must show that the contributions to the limit in (4.2) and (4.3) by all x for which x/n^{γ} lies in the complement of a neighborhood of 0 is exponentially small. The relevant error estimate is given in part (c) of Lemma 4.4. Similar considerations apply to the proofs of the MDPs in section 8, for which the relevant error estimate is given in part (d) of Lemma 4.4.

The steps outlined in the preceding paragraph for deducing the scaling limits of $S_n/n^{1-\gamma}$ from (4.2) and (4.3) are well known in the related contexts of the Curie-Weiss model and the Curie-Weiss-Potts model. Scaling limits for these models are studied in [20, 21] and in [25]

for fixed values of the inverse temperature defining the corresponding canonical ensemble. In contrast to those earlier papers, our study of scaling limits for the BEG model necessitates a considerably more careful analysis because we work with the canonical ensemble P_{n,β_n,K_n}, allowing sequences (β_n, K_n) rather than only fixed values of (β, K).

The analysis of the Taylor expansions of $G_{\beta,K}$ in the neighborhood of a global minimum point involves the notion of the type of a global minimum point, which we next introduce. We temporarily consider any $\beta > 0$ and any $K > 0$ and then specialize to $(\beta, K) \in A \cup B \cup C$. Let $\tilde{z}$ be an element of $\mathcal{E}_{\beta,K}$. Since $G_{\beta,K}$ is real analytic and $\tilde{z}$ is a global minimum point, there exists a positive integer $r = r(\tilde{z})$ such that $G_{\beta,K}^{(2r)}(\tilde{z}) > 0$ and

$$G_{\beta,K}(z) = G_\beta(\tilde{z}) + \frac{G_{\beta,K}^{(2r)}(\tilde{z})}{(2r)!}(z - \tilde{z})^{2r} + O((z - \tilde{z})^{2r+1}) \ \text{ as } z \longrightarrow \tilde{z}.$$

We call $r(\tilde{z})$ the type of the global minimum point $\tilde{z}$. If $r = 1$, then $G_{\beta,K}^{(2)}(\tilde{z}) = 2\beta K - (2\beta K)^2(c_\beta)''(2\beta K\tilde{z})$, and if $r \geq 2$, then $G_\beta^{(2r)}(\tilde{z}) = -(2\beta K)^{2r}c_\beta^{(2r)}(\tilde{z})$.

In Theorem 6.3 in [22] the types of the elements of $\mathcal{E}_{\beta,K}$ are determined for all $\beta > 0$ and $K > 0$. In our study of scaling limits of $S_n/n^{1-\gamma}$ in the present paper, we focus on $(\beta, K) \in A \cup B \cup C$, for which $\mathcal{E}_{\beta,K} = \{0\}$ [Thm. 3.2(a)]. Although the conclusion in [22] that for $(\beta, K) \in B$ the type of 0 equals 2 is correct, the formula for $G_{\beta,K}^{(4)}(0)$ given in (6.6) in that paper has a small error. The correct formula for $G_{\beta,K}^{(4)}(0)$ is given in (4.10) with $(\beta_n, K_n) = (\beta, K)$.

Theorem 4.2. *For all* $(\beta, K) \in A \cup B \cup C$, $\mathcal{E}_{\beta,K} = \{0\}$.
    (a) *For all* $(\beta, K) \in A$, $\tilde{z} = 0$ *has type* $r = 1$.
    (b) *For all* $(\beta, K_c(\beta)) \in B$, $\tilde{z} = 0$ *has type* $r = 2$.
    (c) *For* $(\beta, K) = (\beta_c, K_c(\beta_c)) \in C$, $\tilde{z} = 0$ *has type* $r = 3$.

For all other values of $\beta > 0$ and $K > 0$ not considered in Theorem 4.2, the elements of $\mathcal{E}_{\beta,K}$ all have type $r = 1$. This includes the values $0 < \beta \leq \beta_c$ and $K > K_c(\beta)$ [Thm. 3.2(b)] and the values $\beta > \beta_c$, $K > 0$ [Thm. 3.3]. In these two cases the fact that the elements of $\mathcal{E}_{\beta,K}$ all have type $r = 1$ is proved in [22] in part (c) of Theorem 6.3 and in Theorem 6.4.

We now point out how to prove the breakdown of the law of large numbers stated in (2.4), which holds for $\beta > \beta_c$ and $K = K_c(\beta)$. In this case, $\mathcal{E}_{\beta,K_c(\beta)} = \{0, \pm z(\beta, K)\}$. Since each of the elements of $\mathcal{E}_{\beta,K_c(\beta)}$ has type $r = 1$, the limit in (2.4) is proved exactly as in part (c) of Theorem 2.3 in [25], which treats the breakdown of the law of large numbers for the Curie-Weiss-Potts model at $\beta = \beta_c$. In (2.4),

$$\lambda_0 = \frac{\kappa_0}{\kappa_0 + 2\kappa_1} \ \text{ and } \ \lambda_1 = \frac{\kappa_1}{\kappa_0 + 2\kappa_1}, \tag{4.4}$$

where $\kappa_0 = [G^{(2)}_{\beta,K_c(\beta)}(0)]^{-1/2}$ and $\kappa_1 = [G^{(2)}_{\beta,K_c(\beta)}(z(\beta, K_c(\beta)))]^{-1/2}$.

We return to Lemma 4.1 and in particular to (4.2)–(4.3), which express the scaling limit of $S_n/n^{1-\gamma}$ in terms of the function $nG_{\beta_n,K_n}(x/n^\gamma)$. Using the information about the three different types of the global minimum point of $G_{\beta,K}$ at 0 for $(\beta, K) \in A$, $(\beta, K) \in B$, and $(\beta, K) \in C$, we now indicate the three different forms of the Taylor expansion of $nG_{\beta_n,K_n}(x/n^\gamma)$ needed to deduce the scaling limits of $S_n/n^{1-\gamma}$. These involve the quantities $G^{(2)}_{\beta_n,K_n}(0)$, $G^{(4)}_{\beta_n,K_n}(0)$, and $G^{(6)}_{\beta_n,K_n}(0)$, for the first two of which explicit formulas in terms of β_n and K_n are given. As we will see in later sections, these formulas will guide us into how we should choose the sequences (β_n, K_n) so that all the different scaling limits of $S_n/n^{1-\gamma}$ appear. Since G_{β_n,K_n} is symmetric around 0, all odd-order derivatives of this function evaluated at 0 vanish; in addition, $G_{\beta_n,K_n}(0) = 0$.

In order to state part (d) of the theorem, we define for $\beta > 0$

$$K(\beta) = \frac{1}{2c''_\beta(0)} = \frac{e^\beta + 2}{4\beta}. \tag{4.5}$$

For $0 < \beta \leq \beta_c$ this function coincides with the function $K_c(\beta)$ defined in (3.6), while for $\beta > \beta_c$, $K(\beta) > K_c(\beta)$ [22, Thm. 3.8]. Thus for $(\beta, K) \in B$ we have $K = K_c(\beta) = K(\beta)$ while for $(\beta, K) \in C$ we have $\beta = \beta_c$ and $K = K_c(\beta_c) = K(\beta_c)$.

Theorem 4.3. *Let (β_n, K_n) be any positive bounded sequence and γ any positive number. The following conclusions hold.*

(a) Assume that $(\beta_n, K_n) \to (\beta, K) \in A$. Then the type of $0 \in \mathcal{E}_{\beta,K}$ equals 1. In addition, for any $R > 0$ and for all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$ there exists $\xi = \xi(x/n^\gamma) \in [-x/n^\gamma, x/n^\gamma]$ such that

$$nG_{\beta_n,K_n}(x/n^\gamma) = \frac{1}{n^{2\gamma-1}} \frac{G^{(2)}_{\beta_n,K_n}(0)}{2!}x^2 + \frac{1}{n^{3\gamma-1}}A_n(\xi(x/n^\gamma))x^3. \tag{4.6}$$

The error terms $A_n(\xi(x/n^\gamma))$ are uniformly bounded over $n \in \mathbb{N}$ and $x \in (-Rn^\gamma, Rn^\gamma)$. Furthermore, as $n \to \infty$, $G^{(2)}_{\beta_n,K_n}(0) \to G^{(2)}_{\beta,K}(0) > 0$.

(b) Assume that $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$. Then the type of $0 \in \mathcal{E}_{\beta,K_c(\beta)}$ is 2. In addition, for any $R > 0$ and for all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$ there exists $\xi = \xi(x/n^\gamma) \in [-x/n^\gamma, x/n^\gamma]$ such that

$$nG_{\beta_n,K_n}(x/n^\gamma) = \frac{1}{n^{2\gamma-1}} \frac{G^{(2)}_{\beta_n,K_n}(0)}{2!}x^2 + \frac{1}{n^{4\gamma-1}} \frac{G^{(4)}_{\beta_n,K_n}(0)}{4!}x^4 + \frac{1}{n^{5\gamma-1}}B_n(\xi(x/n^\gamma))x^5. \tag{4.7}$$

The error terms $B_n(\xi(x/n^\gamma))$ are uniformly bounded over $n \in \mathbb{N}$ and $x \in (-Rn^\gamma, Rn^\gamma)$. Furthermore, as $n \to \infty$, $G^{(2)}_{\beta_n,K_n}(0) \to G^{(2)}_{\beta,K_c(\beta)}(0) = 0$ while $G^{(4)}_{\beta_n,K_n}(0) \to G^{(4)}_{\beta,K_c(\beta)}(0) > 0$.

(c) *Assume that* $(\beta_n, K_n) \rightarrow (\beta_c, K_c(\beta_c))$. *Then the type of* $0 \in \mathcal{E}_{\beta_c, K_c(\beta_c)}$ *is 3. In addition, for any* $R > 0$ *and for all* $x \in \mathbb{R}$ *satisfying* $|x/n^\gamma| < R$ *there exists* $\xi = \xi(x/n^\gamma) \in [-x/n^\gamma, x/n^\gamma]$ *such that*

$$nG_{\beta_n, K_n}(x/n^\gamma) = \tag{4.8}$$
$$\frac{1}{n^{2\gamma-1}}\frac{G^{(2)}_{\beta_n, K_n}(0)}{2!}x^2 + \frac{1}{n^{4\gamma-1}}\frac{G^{(4)}_{\beta_n, K_n}(0)}{4!}x^4 + \frac{1}{n^{6\gamma-1}}\frac{G^{(6)}_{\beta_n, K_n}(0)}{6!}x^6 + \frac{1}{n^{7\gamma-1}}C_n(\xi(x/n^\gamma))x^7.$$

The error terms $C_n(\xi(x/n^\gamma))$ *are uniformly bounded over* $n \in \mathbb{N}$ *and* $x \in (-Rn^\gamma, Rn^\gamma)$. *Furthermore, as* $n \rightarrow \infty$, $G^{(2)}_{\beta_n, K_n}(0) \rightarrow G^{(2)}_{\beta_c, K_c(\beta_c)}(0) = 0$ *and* $G^{(4)}_{\beta_n, K_n}(0) \rightarrow G^{(4)}_{\beta_c, K_c(\beta_c)}(0) = 0$ *while* $G^{(6)}_{\beta_n, K_n}(0) \rightarrow G^{(6)}_{\beta_c, K_c(\beta_c)}(0) = 2 \cdot 3^4$.

(d) *For* $\beta > 0$ *we define* $K(\beta)$ *in* (4.5). *Then in* (4.6)–(4.8)

$$G^{(2)}_{\beta_n, K_n}(0) = \frac{2\beta_n K_n(e^{\beta_n} + 2 - 4\beta_n K_n)}{e^{\beta_n} + 2} = \frac{2\beta_n K_n[K(\beta_n) - K_n]}{K(\beta_n)} \tag{4.9}$$

and

$$G^{(4)}_{\beta_n, K_n}(0) = \frac{2(2\beta_n K_n)^4(4 - e^{\beta_n})}{(e^{\beta_n} + 2)^2}. \tag{4.10}$$

Proof. In parts (a), (b), and (c) the type of the global minimum point at 0 is specified in Theorem 4.2. The formulas for $G^{(2)}_{\beta_n, K_n}(0)$ and $G^{(4)}_{\beta_n, K_n}(0)$ in part (d) follow from an explicit calculation of the derivatives and from the formula for $K(\beta)$ given in (4.5). In addition, one evaluates the limits of the Taylor coefficients given in the last sentence of each part (a), (b), and (c) using the continuity of the derivatives $G^{(2j)}_{\beta, K}(0)$ with respect to β and K and the fact that the type of the global minimum point of $G_{\beta, K}$ at 0 is, respectively, $r = 1$, $r = 2$, and $r = 3$.

We now prove the form of the Taylor expansion given in part (c); the forms of the Taylor expansions given in parts (a) and (b) are proved similarly. By Taylor's Theorem, for any $R > 0$ and for all $u \in \mathbb{R}$ satisfying $|u| < R$ there exists $\xi = \xi(u) \in [-u, u]$ such that

$$G_{\beta_n, K_n}(u) = \frac{G^{(2)}_{\beta_n, K_n}(0)}{2!}u^2 + \frac{G^{(4)}_{\beta_n, K_n}(0)}{4!}u^4 + \frac{G^{(6)}_{\beta_n, K_n}(0)}{6!}u^6 + C_n(\xi(u))u^7, \tag{4.11}$$

where $C_n(\xi(u)) = G^{(7)}_{\beta_n, K_n}(\xi(u))/7!$. Because the sequence (β_n, K_n) is positive and bounded, there exists $b \in (0, \infty)$ such that $0 < \beta_n \leq b$ and $0 < K_n \leq b$ for all n. As a continuous function of (β, K, x) on the compact set $[0, b] \times [0, b] \times [-R, R]$, $G^{(7)}_{\beta, K}(x)$ is uniformly bounded. It follows that $G^{(7)}_{\beta_n, K_n}(\xi(u))$, and thus $C_n(\xi(u))$, are uniformly bounded over $n \in \mathbb{N}$ and $u \in (-R, R)$. Multiplying both sides of (4.11) by n and substituting $u = x/n^\gamma$ yields part (c). ∎

This completes our preliminary discussion of the Taylor expansions of $nG_{\beta_n,K_n}(x/n^\gamma)$ as they are needed to deduce the scaling limits of $S_n/n^{1-\gamma}$ via Lemma 4.1. In order to finalize our analysis of these scaling limits, we will have to prove that the contributions to the integrals in (4.2) and (4.3) by $x \in \mathbb{R}$ satisfying $|x/n^\gamma| \geq R$ converge to 0 as $n \to \infty$. In part (c) of the next lemma we prove that the convergence to 0 is exponentially fast. The technical hypothesis in part (c) is satisfied in each of the theorems that proves the scaling limits [Thms. 5.1, 6.1, 7.1]. In part (d) of the next lemma we prove the exponentially fast convergence to 0 of a related integral that arises in the proof of the MDPs. As we verify in the proof of Theorem 8.1, the technical hypothesis in part (d) is satisfied in that setting. The estimates in parts (c) and (d) are consequences of the LDP proved in part (b), which in turn follows from part (a) and the representation formula in Lemma 4.1.

Lemma 4.4 is the main technical innovation in this paper. When adapted to the BEG model, the precursors of Lemma 4.4 given in Lemma 3.5 in [20] and Lemma 3.3 in [25] are able to handle only positive sequences (β_n, K_n) converging to $(\beta, K) \in A \cup B \cup C$ for which G_{β_n,K_n} has a unique global minimum point at 0. In order to handle sequences (β_n, K_n) for which G_{β_n,K_n} has nonunique global minimum points, the modifications that would be necessary in the precursors of Lemma 4.4 would introduce serious technical complications in the proofs of the scaling limits and the MDPs. By allowing us to handle any positive sequence (β_n, K_n) converging to $(\beta, K) \in A \cup B \cup C$, parts (c) and (d) of Lemma 4.4 are universal bounds that enable us to avoid these technical complications altogether.

Lemma 4.4. *Let (β_n, K_n) be any positive sequence converging to $(\beta, K) \in A \cup B \cup C$ and as in Lemma 4.1, let W_n be a sequence of $N(0, (2\beta_n K_n)^{-1})$ random variables defined on a probability space $(\Omega, \mathcal{F}, Q)$. The following conclusions hold.*

(a) There exist $a_1 > 0$ and $a_2 > 0$ such that for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$, $G_{\beta_n,K_n}(x) \geq a_1(|x| - 1)^2 - a_2$.

(b) With respect to $P_{n,\beta_n,K_n} \times Q$, $S_n/n + W_n/n^{1/2}$ satisfies the LDP on $\mathbb{R}$ with exponential speed n and rate function $G_{\beta,K}$.

(c) Given $\gamma > 0$ and $R > 0$, we define

$$y_n = \int_{\{|x| < Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)] \, dx. \tag{4.12}$$

If the sequence y_n is bounded, then there exists $a_3 > 0$ and $a_4 > 0$ such that for all sufficiently large n

$$\int_{\{|x| \geq Rn^\gamma\}} \exp[-n \, G_{\beta_n,K_n}(x/n^\gamma)] \, dx \leq a_3 \exp(-na_4) \to 0.$$

(d) Assume that there exist $\gamma > 0$, $R > 0$, $u \in (0,1)$, $a_5 > 0$, and $a_6 \in \mathbb{R}$ such that for all

sufficiently large n

$$y_n = \int_{\{|x|<Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx \le a_5 \exp(n^u a_6).$$

Then there exists $a_7 > 0$ such that for all sufficiently large n

$$\int_{\{|x|\ge Rn^\gamma\}} \exp[-n\, G_{\beta_n,K_n}(x/n^\gamma)]\, dx \le 2a_5 \exp(-na_7) \to 0.$$

Proof. (a) Because the sequence (β_n, K_n) is bounded and remains a positive distance from the origin and the coordinate axes, there exist $0 < b_1 < b_2 < \infty$ such that $b_1 \le \beta_n \le b_2$ and $b_1 \le K_n \le b_2$ for all $n \in \mathbb{N}$. The conclusion of part (a) is a consequence of the elementary inequalities

$$\begin{aligned} G_{\beta_n,K_n}(x) &=& \beta_n K_n x^2 - c_{\beta_n}(2\beta_n K_n x) \\ &\ge& \beta_n K_n x^2 - 2\beta_n K_n |x| - \log 4 \ge b_1^2(|x|-1)^2 - b_2^2 - \log 4. \end{aligned}$$

(b) We prove that for any bounded, continuous function ψ

$$\lim_{n\to\infty} \frac{1}{n}\log \int_{\Lambda^n \times \Omega} \exp\left[n\psi\left(\frac{S_n}{n} + \frac{W_n}{n^{1/2}}\right)\right] d(P_{n,\beta_n,K_n} \times Q) = \sup_{x\in\mathbb{R}}\{\psi(x) - G_{\beta,K}(x)\}. \quad (4.13)$$

This Laplace principle implies the LDP stated in part (b) [13, Thm. 1.2.3]. $G_{\beta,K}$ is continuous, and by part (a) of this lemma applied to the constant sequence $(\beta_n, K_n) = (\beta, K)$, this function has compact level sets. Since $(\beta, K) \in A \cup B \cup C$, $G_{\beta,K}$ has a unique global minimum point at 0, and therefore $\inf_{x\in\mathbb{R}} G_{\beta,K}(x) = 0$. It follows that $G_{\beta,K}$ is a rate function. We now use Lemma 4.1 with $\gamma = 0$ to rewrite the integral in the last display as

$$\int_{\Lambda^n \times \Omega} \exp\left[n\psi\left(\frac{S_n}{n} + \frac{W_n}{n^{1/2}}\right)\right] d(P_{n,\beta_n,K_n} \times Q) \qquad (4.14)$$

$$= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x)]\, dx} \cdot \int_{\mathbb{R}} \exp[n\{\psi(x) - G_{\beta_n,K_n}(x)\}]\, dx.$$

By part (a) of this lemma, there exist $M > 0$ and $a_8 > 0$ having the following three properties:

1. $G_{\beta_n,K_n}(x) \ge a_8 x^2$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ satisfying $|x| \ge M$.

2. The supremum of $\psi - G_{\beta,K}$ on $\mathbb{R}$ is attained on the interval $[-M, M]$.

3. Let $\Delta = \sup_{x \in \mathbb{R}} \{\psi(x) - G_{\beta,K}(x)\}$. Then $\|\psi\|_\infty - a_8 M^2 \leq -|\Delta| - 1$.

Since G_{β_n, K_n} converges uniformly to $G_{\beta,K}$ on $[-M, M]$, we have for any $\delta > 0$ and all sufficiently large n

$$\exp(-n\delta) \int_{\{|x| \leq M\}} \exp[n\{\psi(x) - G_{\beta,K}(x)\}] \, dx$$
$$\leq \int_{\{|x| \leq M\}} \exp[n\{\psi(x) - G_{\beta_n, K_n}(x)\}] \, dx$$
$$\leq \exp(n\delta) \int_{\{|x| \leq M\}} \exp[n\{\psi(x) - G_{\beta,K}(x)\}] \, dx.$$

In addition, by items 1 and 3

$$\int_{\{|x| > M\}} \exp[n\{\psi(x) - G_{\beta_n, K_n}(x)\}] \, dx$$
$$\leq \exp[n\|\psi\|_\infty] \int_{\{|x| > M\}} \exp[-na_8 x^2] \, dx$$
$$\leq \frac{1}{nMa_8} \exp[n\|\psi\|_\infty - na_8 M^2]$$
$$\leq \frac{1}{nMa_8} \exp[-n(|\Delta| + 1)].$$

We now put these estimates together. For all sufficiently large n we have

$$\exp(-n\delta) \int_{\{|x| \leq M\}} \exp[n\{\psi(x) - G_{\beta,K}(x)\}] \, dx$$
$$\leq \int_{\mathbb{R}} \exp[n\{\psi(x) - G_{\beta_n, K_n}(x)\}] \, dx$$
$$\leq \exp(n\delta) \int_{\{|x| \leq M\}} \exp[n\{\psi(x) - G_{\beta,K}(x)\}] \, dx + \frac{1}{nMa_8} \exp[-n(|\Delta| + 1)].$$

Since by item 2

$$\lim_{n \to \infty} \frac{1}{n} \log \int_{\{|x| \leq M\}} \exp[n\{\psi(x) - G_{\beta,K}(x)\}] \, dx$$
$$= \sup_{\{|x| \leq M\}} \{\psi(x) - G_{\beta,K}(x)\} = \sup_{x \in \mathbb{R}} \{\psi(x) - G_{\beta,K}(x)\},$$

we see that

$$\sup_{x\in\mathbb{R}}\{\psi(x) - G_{\beta,K}(x)\} - \delta$$

$$\leq \liminf_{n\to\infty} \frac{1}{n}\log \int_{\mathbb{R}} \exp[n\{\psi(x) - G_{\beta_n,K_n}(x)\}]\, dx$$

$$\leq \limsup_{n\to\infty} \frac{1}{n}\log \int_{\mathbb{R}} \exp[n\{\psi(x) - G_{\beta_n,K_n}(x)\}]\, dx$$

$$\leq \sup_{x\in\mathbb{R}}\{\psi(x) - G_{\beta,K}(x)\} + \delta,$$

and since $\delta > 0$ is arbitrary, it follows that

$$\lim_{n\to\infty} \frac{1}{n}\log \int_{\mathbb{R}} \exp[n\{\psi(x) - G_{\beta_n,K_n}(x)\}]\, dx = \sup_{x\in\mathbb{R}}\{\psi(x) - G_{\beta,K}(x)\}.$$

We combine this limit with the same limit for $\psi = 0$ and use (4.14) together with the fact that $\inf_{x\in\mathbb{R}} G_{\beta,K}(x) = G_{\beta,K}(0) = 0$, concluding that

$$\lim_{n\to\infty} \frac{1}{n}\log \int_{\Lambda^n\times\Omega} \exp\left[n\psi\left(\frac{S_n}{n} + \frac{W_n}{n^{1/2}}\right)\right] d(P_{n,\beta_n,K_n}\times Q)$$

$$= \sup_{x\in\mathbb{R}}\{\psi(x) - G_{\beta,K}(x)\} - \inf_{x\in R} G_{\beta,K}(x) = \sup_{x\in\mathbb{R}}\{\psi(x) - G_{\beta,K}(x)\}.$$

This is the Laplace principle (4.13). The proof of part (b) is complete.

(c) Since $G_{\beta,K}$ has a unique global minimum point at 0, the LDP proved in part (b) implies the existence of $a_9 > 0$ such that for all $n \in \mathbb{N}$

$$P_{n,\beta_n,K_n}\times Q\left\{\frac{S_n}{n} + \frac{W_n}{n^{1/2}} \notin (-R, R)\right\} \leq \exp(-na_9). \tag{4.15}$$

Using Lemma 4.1, we rewrite the probability in the last display as

$$P_{n,\beta_n,K_n}\times Q\left\{\frac{S_n}{n} + \frac{W_n}{n^{1/2}} \notin (-R, R)\right\} \tag{4.16}$$

$$= P_{n,\beta_n,K_n}\times Q\left\{\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}} \notin (-Rn^\gamma, Rn^\gamma)\right\}$$

$$= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx} \cdot \int_{\{|x|\geq Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx$$

$$= \frac{z_n}{y_n + z_n},$$

where y_n is defined in (4.12) and

$$z_n = \int_{\{|x| \geq Rn^\gamma\}} \exp[-n\, G_{\beta_n, K_n}(x/n^\gamma)]\, dx.$$

Since by hypothesis the sequence y_n is bounded, there exists $y > 0$ such that $y_n \leq y$ for all n. It follows from (4.15) and (4.16) that for all sufficiently large n

$$\tfrac{1}{2}z_n \leq z_n(1 - \exp(-na_9)) \leq y_n \exp(-na_9) \leq y \exp(-na_9)$$

and thus for all sufficiently large n, $z_n \leq 2y \exp(-na_9)$. This completes the proof of part (c).

(d) Exactly as in the proof of part (c), we have for all sufficiently large n

$$\tfrac{1}{2}z_n \leq z_n(1 - \exp(-na_9)) \leq y_n \exp(-na_9).$$

Since by hypothesis $y_n \leq a_5 \exp(n^u a_6)$ and $u \in (0,1)$, it follows that for all sufficiently large n

$$z_n \leq 2a_5 \exp(-na_9 + n^u a_6) \leq 2a_5 \exp(-na_9/2).$$

This completes the proof of part (d). ∎

In the next section we begin our analysis of the scaling limits of $S_n/n^{1-\gamma}$ in the simplest case by considering $(\beta_n, K_n) \to (\beta, K) \in A$. In the two sections following the next one, we will uncover a wider variety of scaling limits by considering sequences (β_n, K_n) converging to $(\beta, K_c(\beta)) \in B$ and to $(\beta_c, K_c(\beta_c)) \in C$.

5   1 Scaling Limit for $(\beta_n, K_n) \to (\beta, K) \in A$

In this short section, we deduce the unique scaling limit of $S_n/n^{1-\gamma}$ when (β_n, K_n) is any positive sequence converging to $(\beta, K) \in A$. The unique global minimum point of $G_{\beta, K}$ at 0 has type $r = 1$ [Thm. 4.2(a)]. As the next theorem shows, the scaling limit with respect to P_{n, β_n, K_n} has the form of a central limit-type theorem that is independent of the particular sequence chosen. In addition, the only value of γ for which $S_n/n^{1-\gamma}$ has a nontrivial limit is $\gamma = 1/2$. We are including this scaling limit in order to highlight the much more complicated behavior of the scaling limits of $S_n/n^{1-\gamma}$ in the subsequent two sections, in which $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$ and $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c)) \in C$ and in which different forms of the limit can be obtained by choosing different sequences.

The following theorem, stated for $0 < \beta \leq \beta_c$ and $0 < K < K_c(\beta)$, is also valid for $\beta > \beta_c$ and $0 < K < K_c(\beta)$, and the proof is essentially the same. The key observation is that for $\beta > \beta_c$, we have $K(\beta) = (e^\beta + 2)/(4\beta) > K_c(\beta)$ [22, Thm. 3.8]. Hence if $K < K_c(\beta)$, then also $K < K(\beta)$ and thus $G_{\beta, K}^{(2)}(0)$ in (5.2) is positive.

Theorem 5.1. *Let (β_n, K_n) be an arbitrary positive sequence that converges to $(\beta, K) \in A$; thus β and K satisfy $0 < \beta \le \beta_c$ and $0 < K < K_c(\beta)$. Then*

$$P_{n,\beta_n,K_n}\{S_n/n^{1/2} \in dx\} \Longrightarrow \exp(-c_2 x^2)\, dx,$$

where $c_2 > 0$ is defined by

$$c_2 = \frac{1}{2} \cdot \frac{1}{[G_{\beta,K}^{(2)}(0)]^{-1} - [2\beta K]^{-1}} = \beta[K(\beta) - K]. \tag{5.1}$$

Thus the limit is independent of the particular sequence (β_n, K_n) that is chosen.

Proof. We use the Taylor expansion in part (a) of Theorem 4.3 with $\gamma = 1/2$. By continuity, $G_{\beta_n,K_n}^{(2)}(0)$ given in (4.9) converges to

$$G_{\beta,K}^{(2)}(0) = \frac{2\beta K[K(\beta) - K]}{K(\beta)}, \tag{5.2}$$

which is positive since $0 < K < K_c(\beta) = K(\beta)$. For any $R > 0$ the error terms $A_n(x/n^{1/2})$ in the Taylor expansion are uniformly bounded over $n \in \mathbb{N}$ and $x \in (-Rn^{1/2}, Rn^{1/2})$. It follows that for all $x \in \mathbb{R}$

$$\lim_{n\to\infty} nG_{\beta_n,K_n}(x/n^{1/2}) = \tfrac{1}{2}G_{\beta,K}^{(2)}(0)x^2$$

and that $R > 0$ can be chosen to be sufficiently small so that for all sufficiently large n and all $x \in \mathbb{R}$ satisfying $|x/n^{1/2}| < R$

$$nG_{\beta_n,K_n}(x/n^{1/2}) \ge \tfrac{1}{4}G_{\beta,K}^{(2)}(0)x^2.$$

Since $\int_{\mathbb{R}} \exp[-G_{\beta,K}^{(2)}(0)x^2/4]dx < \infty$, the dominated convergence theorem implies that for any bounded, continuous function f

$$\lim_{n\to\infty} \int_{\{|x|<Rn^{1/2}\}} f(x)\, \exp[-nG_{\beta_n,K_n}(x/n^{1/2})]\, dx = \int_{\mathbb{R}} f(x)\, \exp[-G_{\beta,K}^{(2)}(0)x^2/2]\, dx.$$

The existence of this limit implies that the sequence $y_n = \int_{\{|x|<Rn^{1/2}\}} \exp[-nG_{\beta_n,K_n}(x/n^{1/2})]dx$ is bounded. Hence, combining this limit with part (c) of Lemma 4.4 yields

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, \exp[-nG_{\beta_n,K_n}(x/n^{1/2})]\, dx = \int_{\mathbb{R}} f(x)\, \exp[-G_{\beta,K}^{(2)}(0)x^2/2]\, dx.$$

We now augment this limit with the same limit for $f = 1$ and use (4.3) to obtain

$$\lim_{n \to \infty} \int_{\Lambda^n \times \Omega} f(S_n/n^{1/2} + W_n) \, d(P_{n,\beta,K_c(\beta)} \times Q)$$

$$= \frac{1}{\int_{\mathbb{R}} \exp[-G_{\beta,K}^{(2)}(0)x^2/2] \, dx} \cdot \int_{\mathbb{R}} f(x) \exp[-G_{\beta,K}^{(2)}(0)x^2/2] \, dx.$$

We omit the straightforward argument using characteristic functions that enables one to deduce from the last display that

$$P_{n,\beta_n,K_n}\{S_n/n^{1/2} \in dx\} \implies \exp(-c_2 x^2) \, dx,$$

where c_2 is given by the first equality in (5.1). A similar argument involving moment generating functions is given on pages 70–71 of [25]. The positivity of c_2 and the second formula for c_2 given in (5.1) follow from (5.2). This completes the proof of the theorem. ∎

In Theorem 8.2 we prove an MDP for $S_n/n^{1-\gamma}$ that is related to the scaling limit proved in Theorem 5.1. As in the latter theorem, the form of the MDP is independent of the particular sequence (β_n, K_n) converging to $(\beta, K) \in A$. In the next section we see the first example of scaling limits for $S_n/n^{1-\gamma}$ where different forms of the limit can be obtained by choosing different sequences $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$.

6   4 Scaling Limits for $(\boldsymbol{\beta_n, K_n}) \to (\boldsymbol{\beta, K_c(\beta)}) \in B$

In this section we determine the scaling limits of $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n}, where (β_n, K_n) is an appropriate positive sequence converging to $(\beta, K_c(\beta)) \in B$ and $\gamma \in (0, 1/2)$ is appropriately chosen. We recall that B is the curve of second-order points for the BEG model. For any $(\beta, K) \in B$, we have $0 < \beta < \beta_c = \log 4$ and

$$K = K_c(\beta) = \frac{1}{2\beta c_\beta''(0)} = \frac{e^\beta + 2}{4\beta}.$$

The scaling limits that we obtain involve limiting densities proportional to $\exp[-G(x)]$, where G takes one of the 4 forms of an even polynomial of degree 4 or 2 satisfying $G(0) = 0$ and $G(x) \to \infty$ as $|x| \to \infty$. There are 3 such G's of degree 4; namely, $G(x) = c_4 x^4$, where $c_4 > 0$ and $G(x) = k\beta x^2 + c_4 x^4$, where $c_4 > 0$ and either $k > 0$ or $k < 0$. There is also 1 such G of degree 2; namely, $G(x) = k\beta x^2$, where $k > 0$. These 4 cases are all obtained in Theorem 6.1; the forms of the limits depend on the choice of $K_n \to K_c(\beta)$ but are independent of the choice of $\beta_n \to \beta$.

In order to determine the forms of the scaling limits of $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n}, we start by recalling the Taylor expansion given in part (b) of Theorem 4.3. For any $\gamma > 0$ and $R > 0$ and for all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$ there exists $\xi \in [-x/n^\gamma, x/n^\gamma]$ such that

$$nG_{\beta_n,K_n}(x/n^\gamma) = \frac{1}{n^{2\gamma-1}}\frac{G^{(2)}_{\beta_n,K_n}(0)}{2!}x^2 + \frac{1}{n^{4\gamma-1}}\frac{G^{(4)}_{\beta_n,K_n}(0)}{4!}x^4 + \frac{1}{n^{5\gamma-1}}B_n(\xi(x/n^\gamma))x^5. \quad (6.1)$$

The error terms $B_n(\xi(x/n^\gamma))$ are uniformly bounded over $n \in \mathbb{N}$ and $x \in (-Rn^\gamma, Rn^\gamma)$. According to part (b) of Theorem 4.2, the unique global minimum point of $G_{\beta,K_c(\beta)}$ at 0 has type 2. Hence by continuity, as $n \to \infty$,

$$G^{(2)}_{\beta_n,K_n}(0) = \frac{2\beta_n K_n[K(\beta_n) - K_n]}{K(\beta_n)} \to G^{(2)}_{\beta,K_c(\beta)}(0) = 0$$

while $G^{(4)}_{\beta_n,K_n}(0) \to G^{(4)}_{\beta,K_c(\beta)}(0) > 0$. We recall that in the last display $K(\beta) = (e^\beta + 2)/(4\beta)$ for $\beta > 0$.

Fixing $\beta \in (0, \beta_c)$, we let β_n be an arbitrary positive sequence that converges to β, and we let θ be a positive number. The key insight is to choose K_n so that $G^{(2)}_{\beta_n,K_n}(0) \to 0$ at a rate $1/n^\theta$, where $1/n^\theta$ counterbalances the term $1/n^{2\gamma-1}$ appearing in (6.1). Since $2\beta_n K_n/K(\beta_n)$ has the positive limit 2β as $n \to \infty$, we achieve this by choosing $k \neq 0$ and defining

$$K_n = K(\beta_n) - k/n^\theta. \quad (6.2)$$

Since $\beta_n \to \beta$ and $K(\cdot)$ is continuous, it follows that $K_n \to K(\beta) = K_c(\beta)$. Hence

$$G^{(2)}_{\beta_n,K_n}(0) = \frac{k}{n^\theta} \cdot \frac{2\beta_n K_n}{K(\beta_n)} = \frac{k}{n^\theta} \cdot C^{(2)}_n, \quad \text{where} \ \ C^{(2)}_n > 0 \ \ \text{and} \ \ C^{(2)}_n \to 2\beta.$$

With these choices (6.1) becomes

$$nG_{\beta_n,K_n}(x/n^\gamma) = \frac{1}{n^{2\gamma+\theta-1}}\frac{kC^{(2)}_n}{2!}x^2 + \frac{1}{n^{4\gamma-1}}\frac{G^{(4)}_{\beta_n,K_n}(0)}{4!}x^4 + \frac{1}{n^{5\gamma-1}}B_n(\xi(x/n^\gamma))x^5. \quad (6.3)$$

As we will see in Theorem 6.1, the scaling limits depend on the value of γ and on K_n through the value of θ, but are independent of the sequence $\beta_n \to \beta$.

In the last display we assume that the coefficients multiplying x^2 and x^4 both appear with nonnegative powers of n and that at least one of these two coefficients has n to the power 0. Then in the limit $n \to \infty$ any coefficient including the error term that has a positive power of n will vanish while any coefficient that has n to the power 0 will converge to a positive constant.

This preliminary analysis shows the possibility of multiple scaling limits for different choices of γ and θ. In order to confirm this possibility, we define

$$v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$$

and focus on the cases in which $v = 0$. As we will see in the final section of the paper, $v < 0$ corresponds to 4 different MDPs for $S_n/n^{1-\gamma}$. On the other hand, if $v > 0$, then one obtains neither scaling limits nor MDPs.

In the next theorem we show that $v = 0$ corresponds to 3 different choices of γ and θ, which in turn correspond to 4 different sequences K_n in (6.2). The additional sequence arises because when x^2 is not the highest order term in the scaling limit (cases 3–4), k can be chosen to be either positive or negative. As shown in Table 6.1 in part (b) of the theorem, for each of these 4 different sequences we obtain 4 different scaling limits of $S_n/n^{1-\gamma}$. In case 1 we can also choose k to be any real number; this affects only the definition of the sequence K_n, not the form of the scaling limit.
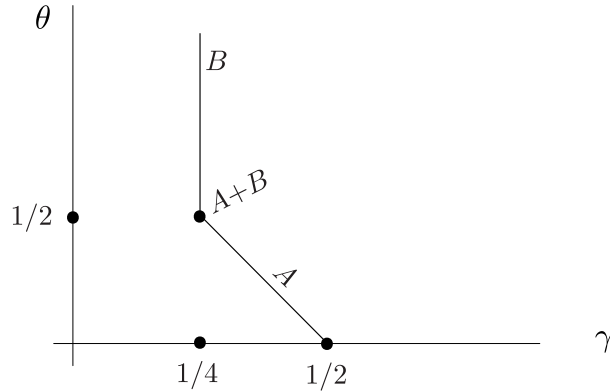

Figure 4: Influence of A and B on scaling limits when $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$

The results of the theorem confirm one's intuition concerning the influence of the regions on the scaling limits. Of the 4 cases, case 1 corresponds to the largest values of θ — namely, $\theta > 1/2$ — and thus the most rapid convergence of $K_n \to K_c(\beta)$. In this case only B influences the form of the limiting density, which is proportional to $\exp(-c_4 x^4)$; c_4 defined in (6.5) is positive since $e^\beta < e^{\beta_c} = 4$. By contrast, case 2 corresponds to the smallest values of θ — namely, $\theta \in (0, 1/2)$ — and thus the slowest convergence of $K_n \to K_c(\beta)$. In this case only A influences the form of the limiting density, which is proportional to $\exp(-\beta x^2)$; thus we have $S_n/n^{1-\gamma}$ converging in distribution to a normal random variable even though the non-classical scaling is given by $n^{1-\gamma}$, where $\gamma = (1 - \theta)/2 \in (1/4, 1/2)$. Finally, cases 3 and 4 correspond

to the critical speed $\theta = 1/2$. In this case both A and B influence the form of the limiting density, which is proportional to $\exp(-k\beta x^2 - c_4 x^4)$ with $c_4 > 0$ and either $k > 0$ or $k < 0$. In Figure 4 we indicate the subsets of the positive quadrant of the θ-γ plane leading to the 4 cases just discussed. Using Table 5.1, one easily checks that as θ increases through the critical value $1/2$, the values of γ in the scaling limit change continuously while the forms of the limiting densities change discontinuously.

Theorem 6.1. *For fixed $\beta \in (0, \beta_c)$, let β_n be an arbitrary positive sequence that converges to β. Given $\theta > 0$ and $k \neq 0$, define*

$$K_n = K(\beta_n) - k/n^\theta,$$

where $K(\beta) = (e^\beta + 2)/(4\beta)$ for $\beta > 0$. Then $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$. Given $\gamma \in (0, 1)$, we also define

$$G(x) = \delta(v, 2\gamma + \theta - 1)k\beta x^2 + \delta(v, 4\gamma - 1)c_4 x^4, \tag{6.4}$$

where $\delta(a, b)$ equals 1 if $a = b$ and equals 0 if $a \neq b$ and $c_4 > 0$ is given by

$$c_4 = \frac{G^{(4)}_{\beta, K_c(\beta)}(0)}{4!} = \frac{2[2\beta K_c(\beta)]^4(4 - e^\beta)}{4!(e^\beta + 2)^2} = \frac{(e^\beta + 2)^2(4 - e^\beta)}{2^3 \cdot 4!}. \tag{6.5}$$

The following conclusions hold.
   (a) *Assume that $v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$ equals 0. Then*

$$P_{n, \beta_n, K_n}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow \exp[-G(x)]\, dx. \tag{6.6}$$

   (b) *We have $v = 0$ if and only if one of the 4 cases enumerated in Table* 6.1 *holds. Each of the 4 cases corresponds to a set of values of θ and γ, to the influence of one or more sets B and A, and to a particular scaling limit in* (6.6). *In case 1 the choice of $k \in \mathbb{R}$ does not affect the form of the scaling limit.*

case influence	values of θ	values of γ	scaling limit of $S_n/n^{1-\gamma}$
1 B	$\theta > \frac{1}{2}$	$\gamma = \frac{1}{4}$	$\exp(-c_4 x^4)\, dx$ $c_4 > 0,\ k \in \mathbb{R}$
2 A	$\theta \in (0, \frac{1}{2})$	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{4}, \frac{1}{2})$	$\exp(-k\beta x^2)\, dx$ $k > 0$
3–4 $A + B$	$\theta = \frac{1}{2}$	$\gamma = \frac{1}{4}$	$\exp(-k\beta x^2 - c_4 x^4)\, dx$ $k \neq 0$

Table 6.1: Values of θ and γ and scaling limits in part (b) of Theorem 6.1

Note. Let $\beta_n = \beta$ for all n. The constant sequence $(\beta_n, K_n) = (\beta, K_c(\beta))$ for all n corresponds to the choice $\theta = \infty$ in case 1. As in the proof of case 1, one shows that $P_{n,\beta,K_c(\beta)}\{S_n/n^{1-1/4} \in dx\} \implies \exp(-c_4 x^4)dx$. This scaling limit was mentioned in (2.6).

Proof of Theorem 6.1. We first prove part (b) assuming part (a), and then we prove part (a).

(b) $v = \min\{2\gamma+\theta-1, 4\gamma-1\}$ equals 0 if and only if each of the quantities in this minimum is nonnegative and one or more of the quantities equals 0. As (6.4) makes clear, $4\gamma - 1 = 0$ corresponds to the influence of B and $2\gamma + \theta - 1 = 0$ to the influence of A. We have the following 4 mutually exclusive and exhaustive cases, which correspond to the 4 cases in Table 6.1.

- **Case 1: Influence of B alone.** $2\gamma + \theta - 1 > 0$, $4\gamma - 1 = 0$, and $k \in \mathbb{R}$. In this case $\gamma = 1/4$ and $\theta > 1 - 2\gamma = 1/2$, which corresponds to the second and third columns for case 1 in Table 6.1.

- **Case 2: Influence of A alone.** $2\gamma+\theta-1 = 0$, $4\gamma-1 > 0$, and $k > 0$. In this case $\gamma > 1/4$ and $\theta = 1 - 2\gamma < 1/2$. Since θ must be positive, we have $\gamma = (1 - \theta)/2 \in (1/4, 1/2)$. Hence case 2 corresponds to the second and third columns for case 2 in Table 6.1.

- **Cases 3–4: Influence of A and B.** $2\gamma + \theta - 1 = 0$, $4\gamma - 1 = 0$, $k > 0$ for case 3, and $k < 0$ for case 4. In these 2 cases $\gamma = 1/4$ and $\theta = 1 - 2\gamma = 1/2$, which corresponds to the second and third columns for cases 3 and 4 in Table 6.1.

In cases 1, 2, 3, and 4 we have, respectively, $G(x) = c_4 x^4$, $G(x) = k\beta x^2$ with $k > 0$, $G(x) = k\beta x^2 + c_4 x^4$ with $k > 0$, and $G(x) = k\beta x^2 + c_4 x^4$ with $k < 0$. In combination with part (a), we obtain the 4 forms of the scaling limits listed in the last column of Table 6.1.

(b) We prove the 4 scaling limits corresponding to the 4 cases listed in Table 6.1. As the discussion prior to the statement of the theorem indicates, the quantity $v = \min\{2\gamma+\theta-1, 4\gamma-1\}$ is defined in such a way that in each of the 4 cases defined by the choices of θ, γ, and k in Table 6.1, we have for each $x \in \mathbb{R}$

$$\lim_{n\to\infty} nG_{\beta_n,K_n}(x/n^\gamma) = G(x).$$

Since in each case we have $\gamma \in [1/4, 1/2)$, the term $W_n/n^{1/2-\gamma}$ in (4.1) does not contribute to the limit $n \to \infty$. Hence we can determine the scaling limits of $S_n/n^{1-\gamma}$ by using (4.2). In order to justify taking the limit inside the integrals on the right hand side of (4.2), we return to (6.3) and use the fact that for all sufficiently large n, $C_n^{(2)} > 0$ and $G_{\beta_n,K_n}^{(4)}(0) > 0$. It follows that $R > 0$ can be chosen to be sufficiently small so that for all sufficiently large n and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$ there exists a polynomial $H(x)$ satisfying

$$nG_{\beta_n,K_n}(x/n^\gamma) \geq H(x) \tag{6.7}$$

and $\int_{\mathbb{R}} \exp[-H(x)]dx < \infty$. In case 1 when $k \geq 0$ as well as in cases 2 and 3, $H(x) = G(x)/2$; in case 1 when $k < 0$ and in case 4, which corresponds to $k < 0$,

$$H(x) = -2|k|\beta x^2 + c_4 x^4/2. \tag{6.8}$$

The last two displays in combination with the dominated convergence theorem imply that for any bounded, continuous function f

$$\lim_{n\to\infty} \int_{\{|x|<Rn^\gamma\}} f(x)\,\exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\,dx = \int_{\mathbb{R}} f(x)\,\exp[-G(x)]\,dx.$$

The existence of this limit implies that the sequence $y_n = \int_{\{|x|<Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]dx$ is bounded. Hence, combining this limit with part (c) of Lemma 4.4 yields

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\,\exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\,dx = \int_{\mathbb{R}} f(x)\,\exp[-G(x)]\,dx.$$

If we augment this limit with the same limit for $f = 1$ and use (4.2), then we conclude that in each of the 4 cases

$$\lim_{n\to\infty} \int_{\Lambda^n} f(S_n/n^{1-\gamma})\,dP_{n,\beta_n,K_n} = \frac{1}{\int_{\mathbb{R}} \exp[-G(x)]\,dx} \cdot \int_{\mathbb{R}} f(x)\,\exp[-G(x)]\,dx.$$

This yields the scaling limits in part (a). The proof of the theorem is complete. ∎

    This finishes our analysis of scaling limits for $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n}, where the sequence (β_n, K_n) converging to $(\beta, K_c(\beta)) \in B$ is defined in Theorem 6.1. This analysis is a warm-up for the even more interesting analysis of the scaling limits for sequences (β_n, K_n) converging to the tricritical point.

7   13 Scaling Limits for $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$

In Theorem 6.1 we obtained 4 forms of scaling limits for $S_n/n^{1-\gamma}$ using sequences (β_n, K_n) converging to a second-order point $(\beta, K_c(\beta)) \in B$. The limiting densities are proportional to $\exp[-G(x)]$, where G takes of the 4 forms of an even polynomial of degree 4 or 2 satisfying $G(0) = 0$ and $G(x) \to \infty$ as $|x| \to \infty$. In each case the form of the limit is independent of the choice of $\beta_n \to \beta$ but depends on the choice of $K_n \to K_c(\beta)$. Like the BEG model at $(\beta, K_c(\beta)) \in B$, the Curie-Weiss model has a second-order phase transition at a critical inverse temperature $\bar{\beta}_c$. The 4 scaling limits and the 4 MDPs analyzed in Theorem 8.1 are analogous to

the scaling limits and MDPs that hold in the Curie-Weiss model when the inverse temperature converges to $\bar{\beta}_c$ along appropriate sequences β_n [14]. However, the 13 scaling limits proved in the present section and the 13 analogous MDPs obtained in Theorem 8.3 depend on the nature of the tricritical point, a feature not shared with the Curie-Weiss model.

We now use the insights gained in the preceding section to study the more complicated problem of scaling limits for $S_n/n^{1-\gamma}$ using sequences (β_n, K_n) converging to the tricritical point $(\beta_c, K_c(\beta_c)) = (\log 4, 3/[2 \log 4])$. As in the preceding section, we choose $\theta > 0$, $k \neq 0$, and

$$K_n = K(\beta_n) - k/n^\theta, \tag{7.1}$$

where $K(\beta) = (e^\beta + 2)/(4\beta)$ for $\beta > 0$. In contrast to the preceding section, we now also have to pick the sequence β_n appropriately. Theorem 7.1 shows that 13 scaling limits arise for different choices of θ, γ, and the parameter appearing in the definition of β_n. The limiting densities are proportional to $\exp[-G(x)]$, where G takes one of the 13 forms of an even polynomial of degree 6, 4, or 2 satisfying $G(0) = 0$ and $G(x) \to \infty$ as $|x| \to \infty$.

In order to determine the forms of the scaling limits for $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n}, we use the Taylor expansion given in part (c) of Theorem 4.3. For any $\gamma > 0$ and $R > 0$ and for all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$ there exists $\xi \in [-x/n^\gamma, x/n^\gamma]$ such that

$$nG_{\beta_n,K_n}(x/n^\gamma) = \tag{7.2}$$
$$\frac{1}{n^{2\gamma-1}} \frac{G^{(2)}_{\beta_n,K_n}(0)}{2!} x^2 + \frac{1}{n^{4\gamma-1}} \frac{G^{(4)}_{\beta_n,K_n}(0)}{4!} x^4 + \frac{1}{n^{6\gamma-1}} \frac{G^{(6)}_{\beta_n,K_n}(0)}{6!} x^6 + \frac{1}{n^{7\gamma-1}} C_n(\xi(x/n^\gamma)) x^7.$$

The error terms $C_n(\xi(x/n^\gamma))$ are uniformly bounded over $n \in \mathbb{N}$ and $x \in (-Rn^\gamma, Rn^\gamma)$. According to part (c) of Theorem 4.2, the unique global minimum point of $G_{\beta_c,K_c(\beta_c)}$ at 0 has type 3. Hence by continuity, as $n \to \infty$,

$$G^{(2)}_{\beta_n,K_n}(0) = \frac{2\beta_n K_n [K(\beta_n) - K_n]}{K(\beta_n)} \to G^{(2)}_{\beta_c,K_c(\beta_c)}(0) = 0,$$

$$G^{(4)}_{\beta_n,K_n}(0) = \frac{2(2\beta_n K_n)^4 (4 - e^{\beta_n})}{(e^{\beta_n} + 2)^2} \to G^{(4)}_{\beta_c,K_c(\beta_c)}(0) = 0,$$

while $G^{(6)}_{\beta_n,K_n}(0) \to G^{(6)}_{\beta_c,K_c(\beta_c)}(0) = 2 \cdot 3^4$.

As in the preceding section, we choose K_n as in (7.1) so that $G^{(2)}_{\beta_n,K_n}(0) \to 0$ at a rate $1/n^\theta$, where $1/n^\theta$ counterbalances the term $1/n^{2\gamma-1}$ appearing in (7.2). We also choose β_n so that $G^{(4)}_{\beta_n,K_n}(0) \to 0$ at a rate $1/n^\alpha$, where $1/n^\alpha$ counterbalances the term $1/n^{4\gamma-1}$ appearing in (7.2). This is achieved by choosing $\alpha > 0$ and either $b > 0$ or $b < 0$ and then defining β_n by the logarithmic formula

$$\beta_n = \log(4 - b/n^\alpha) = \log(e^{\beta_c} - b/n^\alpha); \tag{7.3}$$

if $b > 0$, then β_n is well defined for all sufficiently large n. Since $\beta_n \to \beta$ and $K(\cdot)$ is continuous, it follows that $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$. With this choice of (β_n, K_n) we have

$$G^{(2)}_{\beta_n, K_n}(0) = \frac{k}{n^\theta} \cdot \frac{2\beta_n K_n}{K(\beta_n)} = \frac{k}{n^\theta} \cdot C_n^{(2)}, \quad \text{where} \ \ C_n^{(2)} \to 2\beta_c, \tag{7.4}$$

and

$$G^{(4)}_{\beta_n, K_n}(0) = \frac{b}{n^\alpha} \cdot \frac{2(2\beta_n K_n)^4}{(e^{\beta_n} + 2)^2} = \frac{b}{n^\alpha} \cdot C_n^{(4)}, \quad \text{where} \ \ C_n^{(4)} \to \frac{2(2\beta_c K_c(\beta_c))^4}{(e^{\beta_c} + 2)^2} = \frac{9}{2} > 0. \tag{7.5}$$

The dependence of (β_n, K_n) in (7.1) and (7.3) upon α and θ is complicated; because β_n is a function of α, K_n is both a function of θ and, through β_n, a function of α. However, the α and θ decouple nicely when (7.4) and (7.5) are substituted into (7.2), yielding

$$nG_{\beta_n, K_n}(x/n^\gamma) \tag{7.6}$$
$$= \frac{1}{n^{2\gamma+\theta-1}} \frac{kC_n^{(2)}}{2!} x^2 + \frac{1}{n^{4\gamma+\alpha-1}} \frac{bC_n^{(4)}}{4!} x^4 + \frac{1}{n^{6\gamma-1}} \frac{G^{(6)}_{\beta_n, K_n}(0)}{6!} x^6 + \frac{1}{n^{7\gamma-1}} C_n(\xi(x/n^\gamma)) x^7.$$

We continue the analysis as in the preceding section. Let us suppose that in the last display the coefficients multiplying x^2, x^4, and x^6 all appear with nonnegative powers of n and that at least one of the coefficients has n to the power 0. Then in the limit $n \to \infty$ any coefficient including the error term that has a positive power of n will vanish while any coefficient that has n to the power 0 will converge to positive constants. In order to analyze the various cases, we define

$$w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}, \tag{7.7}$$

and focus on the cases in which $w = 0$. As we will see in the final section of the paper, $w < 0$ corresponds to 13 different MDPs for $S_n/n^{1-\gamma}$. On the other hand, if $w > 0$, then one obtains neither scaling limits nor MDPs.

In the next theorem we show that $w = 0$ corresponds to 7 different choices of γ, θ, and α, which in turn correspond to 13 different sequences (β_n, K_n) defined in (7.1) and (7.3). The additional sequences arise because when x^4 is not the highest order term in the scaling limit (cases 4–5, 8–13), b can be chosen to be either positive or negative; similarly, when x^2 is not the highest order term in the scaling limit (cases 6–13), k can be chosen to be either positive or negative. As shown in Table 7.1 in part (b) of the theorem, for each of these 13 different sequences we obtain a different scaling limit of $S_n/n^{1-\gamma}$.

The limiting densities in cases 1, 4–7, and 10–13 are new. In cases 2, 3b, 8, and 9 we obtain the same forms of the limiting densities as in Theorem 6.1, where we considered $(\beta_n, K_n) \to (\beta, K) \in B$. However, the values of γ in the corresponding scaling limits in the two theorems

are different. By contrast, the values of γ and θ as well as the forms of the limiting densities are the same in case 3a in Theorem 7.1 and in case 2 in Theorem 6.1.

There are yet further possibilities concerning the sign of b and k. In all the cases in which no x^4 term appears in the scaling limit (cases 1, 3, 6, 7), we can choose b to be any real number. Similarly, in all the cases in which no x^2 term appears in the scaling limit (cases 1, 2, 4, 5), we can choose k to be any real number. Although the choice of b or k affects the definition of the sequence (β_n, K_n), it does not affect the form of the scaling limit.

Through the terms x^6, x^4, and x^2 appearing in the limiting densities, the scaling limits correspond to the influence of one or more of the sets C, B, and A. The influence of the various sets upon the form of the scaling limits is shown in Figure 2 in the introduction, and details are given in Table 7.1, which is included in part (b) of the next theorem. Case 3, which corresponds to the influence of A alone, has two subcases, labeled 3a and 3b in Table 7.1. Case 3a corresponds to the lower region labeled A in Figure 2 and case 3b to the upper region labeled A in Figure 2. Using Table 7.1, one easily checks that as (α, θ) crosses any of the lines in Figure 2 labeled $A + B$, $A + C$, or $B + C$, the values of γ in the scaling limits change continuously while the forms of the limiting densities change discontinuously.

Theorem 7.1. *Given $\alpha > 0$, $\theta > 0$, $b \neq 0$, and $k \neq 0$, define*

$$\beta_n = \log(4 - b/n^\alpha) = \log(e^{\beta_c} - b/n^\alpha) \ \text{ and } \ K_n = K(\beta_n) - k/n^\theta,$$

where $K(\beta) = (e^\beta + 2)/(4\beta)$ for $\beta > 0$. Then $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$. Given $\gamma \in (0, 1)$, we also define

$$G(x) = \delta(w, 2\gamma + \theta - 1)k\beta_c x^2 + \delta(w, 4\gamma + \alpha - 1)b\bar{c}_4 x^4 + \delta(w, 6\gamma - 1)c_6 x^6, \qquad (7.8)$$

where $\bar{c}_4 = 3/16$ and $c_6 = 9/40$. The following conclusions hold.
    (a) *Assume that $w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}$ equals 0. Then*

$$P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow \exp[-G(x)]\, dx. \qquad (7.9)$$

(b) *We have $w = 0$ if and only if one of the 13 cases enumerated in Table 7.1 holds. Each of the 13 cases corresponds to a set of values of θ, α, and γ, to the influence of one or more sets C, B, A, and to a particular scaling limit in (7.9). The form of the scaling limit is not affected by the choice of $b \in \mathbb{R}$ in cases 1, 3, 6, and 7 and by the choice of $k \in \mathbb{R}$ in cases 1, 2, 4, and 5.*

case	values of α	values of γ	scaling limit of $S_n/n^{1-\gamma}$
influence	values of θ		
1	$\alpha > \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-c_6 x^6)\,dx$
C	$\theta > \frac{2}{3}$		$c_6 > 0,\, b \in \mathbb{R},\, k \in \mathbb{R}$
2	$\alpha \in (0, \frac{1}{3})$	$\gamma = \frac{1-\alpha}{4} \in (\frac{1}{6}, \frac{1}{4})$	$\exp(-b\bar{c}_4 x^4)\,dx$
B	$\theta > \frac{\alpha+1}{2}$		$\bar{c}_4 > 0,\, b > 0,\, k \in \mathbb{R}$
3a	$\alpha > 0$	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{4}, \frac{1}{2})$	$\exp(-k\beta_c x^2)\,dx$
A	$\theta \in (0, \frac{1}{2})$		$k > 0,\, b \in \mathbb{R}$
3b	$\theta \in [\frac{1}{2}, \frac{2}{3})$	$\gamma = \frac{1-\theta}{2} \in (\frac{1}{6}, \frac{1}{4}]$	$\exp(-k\beta_c x^2)\,dx$
A	$\alpha > 2\theta - 1$		$k > 0,\, b \in \mathbb{R}$
4–5	$\alpha = \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-b\bar{c}_4 x^4 - c_6 x^6)\,dx$
$B + C$	$\theta > \frac{2}{3}$		$b \neq 0,\, k \in \mathbb{R}$
6–7	$\alpha > \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-k\beta_c x^2 - c_6 x^6)\,dx$
$A + C$	$\theta = \frac{2}{3}$		$k \neq 0,\, b \in \mathbb{R}$
8–9	$\alpha \in (0, \frac{1}{3})$	$\gamma = \frac{1-\alpha}{4} \in (\frac{1}{6}, \frac{1}{4})$	$\exp(-k\beta_c x^2 - b\bar{c}_4 x^4)\,dx$
$A + B$	$\theta = \frac{\alpha+1}{2} \in (\frac{1}{2}, \frac{2}{3})$		$k \neq 0,\, b > 0$
10–13	$\alpha = \frac{1}{3}$	$\gamma = \frac{1}{6}$	$\exp(-k\beta_c x^2 - b\bar{c}_4 x^4 - c_6 x^6)\,dx$
$A + B + C$	$\theta = \frac{2}{3}$		$k \neq 0,\, b \neq 0$

Table 7.1: Values of α, θ, and γ and scaling limits in part (b) of Theorem 7.1

Note. The constant sequence $(\beta_n, K_n) = (\beta_c, K_c(\beta_c))$ for all n corresponds to the choices $\alpha = \theta = \infty$ in case 1. As in the proof of case 1, one shows that $P_{n,\beta_c,K_c(\beta_c)}\{S_n/n^{1-1/6} \in dx\} \Longrightarrow \exp(-c_6 x^6)dx$. This scaling limit was mentioned in (2.7).

Proof of Theorem 7.1. We first prove part (b) from part (a) and then prove part (a).

(b) $w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}$ equals 0 if and only if each of the quantities in this minimum is nonnegative and one or more of the quantities equals 0. As (7.8) makes clear, $6\gamma - 1 = 0$ corresponds to the influence of C, $4\gamma + \alpha - 1 = 0$ to the influence of B, and $2\gamma + \theta - 1 = 0$ to the influence of A. We have the following 13 mutually exclusive and exhaustive cases, which correspond to the 13 cases in Table 7.1. In each of the cases the equalities and inequalities expressing the influence of one or more sets C, B, and A are easily verified to be equivalent to the equalities and inequalities involving α, θ, and γ given in the second and third columns of Table 7.1. Case 3, the most complicated, divides into two subcases depending on the value of α.

- **Case 1: Influence of C alone.** $2\gamma + \theta - 1 > 0$, $4\gamma + \alpha - 1 > 0$, $6\gamma - 1 = 0$, $b \in \mathbb{R}$, and

$k \in \mathbb{R}$.

- **Case 2: Influence of B alone.** $2\gamma + \theta - 1 > 0$, $4\gamma + \alpha - 1 = 0$, $6\gamma - 1 > 0$, $b > 0$, and $k \in \mathbb{R}$.

- **Case 3: Influence of A alone.** $2\gamma + \theta - 1 = 0$, $4\gamma + \alpha - 1 > 0$, $6\gamma - 1 > 0$, $k > 0$, and $b \in \mathbb{R}$.

- **Cases 4–5: Influence of B and C.** $2\gamma + \theta - 1 > 0$, $4\gamma + \alpha - 1 = 0$, $6\gamma - 1 = 0$, $b > 0$ for case 4 and $b < 0$ for case 5, and $k \in \mathbb{R}$.

- **Cases 6–7: Influence of A and C.** $2\gamma + \theta - 1 = 0$, $4\gamma + \alpha - 1 > 0$, $6\gamma - 1 = 0$, $k > 0$ for case 6 and $k < 0$ for case 7, and $b \in \mathbb{R}$.

- **Cases 8–9: Influence of A and B.** $2\gamma + \theta - 1 = 0$, $4\gamma + \alpha - 1 = 0$, $6\gamma - 1 > 0$, $k > 0$ for case 8, $k < 0$ for case 9, and $b > 0$.

- **Cases 10–13: Influence of A , B, and C.** $2\gamma + \theta - 1 = 0$, $4\gamma + \alpha - 1 = 0$, $6\gamma - 1 = 0$, $k > 0$ and $b > 0$ for case 10, $k < 0$ and $b > 0$ for case 11, $k > 0$ and $b < 0$ for case 12, and $k < 0$ and $b < 0$ for case 13.

In each of the 13 cases the form of $G(x)$ follows from (7.8). In combination with part (a), we obtain the 13 forms of the scaling limits listed in the last column of Table 7.1.

(a) The proof of the 13 scaling limits follows precisely the pattern of the proof of the 4 scaling limits listed in part (b) of Theorem 6.1. As the discussion preceding the statement of Theorem 7.1 indicates, the quantity $w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}$ is defined in such a way that in each of the 13 cases defined by the choices of α, θ, γ, k, and b in Table 7.1, we have for each $x \in \mathbb{R}$

$$\lim_{n \to \infty} n G_{\beta_n, K_n}(x/n^\gamma) = G(x).$$

Since in each case we have $\gamma \in [1/6, 1/2)$, the term $W_n/n^{1/2-\gamma}$ in (4.1) does not contribute to the limit $n \to \infty$. Hence we can determine the scaling limits of $S_n/n^{1-\gamma}$ by using (4.2). In order to justify taking the limit inside the integrals on the right hand side of (4.2), we return to (7.6) and use the fact that for all sufficiently large n, $C_n^{(2)} > 0$, $C_n^{(4)} > 0$, and $G_{\beta_n, K_n}^{(6)}(0) > 0$. It follows that $R > 0$ can be chosen to be sufficiently small so that for all sufficiently large n and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$ there exists a polynomial $H(x)$ satisfying

$$n G_{\beta_n, K_n}(x/n^\gamma) \geq H(x) \tag{7.10}$$

and $\int_{\mathbb{R}} \exp[-H(x)] < \infty$. We define $H(x) = G(x)/2$ in all the cases in which both $b \geq 0$ and $k \geq 0$ (cases 1–4, 6, 8, 10). Otherwise, a suitable polynomial H can be found as in (6.8); the

details are omitted. As in the proof of Theorem 6.1, the dominated convergence theorem and part (c) of Lemma 4.4 imply that for any bounded, continuous function f

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\, dx = \int_{\mathbb{R}} f(x)\, \exp[-G(x)]\, dx.$$

From (4.2) we conclude that in each of the 13 cases in part (b)

$$P_{n,\beta_n,K_n}\big\{S_n/n^{1-\gamma} \in dx\big\} \Longrightarrow \exp[-G(x)]\, dx.$$

This completes the proof of the theorem. ∎

Two special cases of the scaling limits in Theorem 7.1 are worth pointing out. Given $\theta > 0$ and $k \neq 0$, the sequence

$$\beta_n = \beta_c \ \text{ and } \ K_n = K(\beta_c) - k/n^\theta$$

corresponds to the choice $\alpha = \infty$ in Theorem 7.1. With this sequence and with the same proofs, one obtains exactly the same limits as in cases 1, 3, 6, and 7 in this theorem with the same choices of θ, γ, and k. Similarly, given $\alpha > 0$ and $b \neq 0$, the sequence

$$\beta_n = \log(4 - b/n^\alpha) \ \text{ and } \ K_n = K(\beta_c)$$

corresponds to the choice $\theta = \infty$ in Theorem 7.1. With this sequence and with the same proofs, one obtains exactly the same limits as in cases 1, 2, 4, and 5 in this theorem with the same choices of α, γ, and b.

This completes our analysis of scaling limits for $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n}, where the sequence (β_n, K_n) converging to $(\beta_c, K_c(\beta_c))$ is defined in Theorem 7.1. In the next section we study MDPs for $S_n/n^{1-\gamma}$ for appropriate sequences (β_n, K_n) converging to $(\beta, K) \in A \cup B \cup C$ and for appropriate choices of γ. We obtain 1 MDP for $(\beta, K) \in A$, 4 MDPs for $(\beta, K_c(\beta)) \in B$, and 13 MDPs for $(\beta_c, K_c(\beta_c)) \in C$.

8   18 MDPs for $(\beta_n, K_n) \to (\beta, K) \in A \cup B \cup C$

In this section we turn to a new problem, which is to formulate MDPs for $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n}, first for appropriate sequences (β_n, K_n) converging to $(\beta, K_c(\beta)) \in B$, then for (β_n, K_n) converging to $(\beta, K) \in A$, and finally for (β_n, K_n) converging to $(\beta_c, K_c(\beta_c)) \in C$. These results are stated, respectively, in Theorem 8.1, Theorem 8.2, and Theorem 8.3. In proving the first result, we introduce the methods that are also used to prove the third. The proof of the MDP when $(\beta_n, K_n) \to (\beta, K) \in A$ proceeds differently from the proofs of

the other MDPs in this section, relying on the Gärtner-Ellis Theorem. After the proof of that MDP, we will remark on why the same method cannot be used to prove all the MDPs in this section. Although an MDP is an LDP, we shall follow the example of [14], who in their study of Curie-Weiss-type models speak about an MDP whenever the exponential speed a_n of the large deviation probabilities satisfies $a_n/n \to 0$ as $n \to \infty$. Also see [12, §3.7].

When $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B \cup C$, we will prove the MDPs by a method that is closely related to the proofs of the scaling limits earlier in this paper. Thus, rather than focus on the large deviation probabilities directly, we prove that $S_n/n^{1-\gamma}$ satisfies an equivalent Laplace principle. Despite the similarity in the proof of the scaling limits and the Laplace principles, the proof of the latter is much more delicate, requiring additional estimates not needed in the proof of the former.

We start by considering the MDPs when (β_n, K_n) converges to $(\beta, K_c(\beta)) \in B$. In order to formulate these limit theorems, we adapt the methods used in section 6, where we proved scaling limits for such sequences (β_n, K_n). For $\beta \in (0, \beta_c)$ let β_n be an arbitrary positive sequence that converges to β. Given $\theta > 0$ and $k \neq 0$, we then define $K_n \to K_c(\beta)$ as in (6.2). With this choice, part (b) of Theorem 4.3 implies that for any $\gamma > 0$ and $R > 0$ and for all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$ there exists $\xi \in [-x/n^\gamma, x/n^\gamma]$ such that [see (6.3)]

$$nG_{\beta_n, K_n}(x/n^\gamma) = \frac{1}{n^{2\gamma+\theta-1}} \frac{C_n^{(2)}}{2!} x^2 + \frac{1}{n^{4\gamma-1}} \frac{G_{\beta_n, K_n}^{(4)}(0)}{4!} x^4 + \frac{1}{n^{5\gamma-1}} B_n(\xi(x/n^\gamma))x^5. \quad (8.1)$$

The error terms $B_n(\xi(x/n^\gamma))$ are uniformly bounded over $n \in \mathbb{N}$ and $x \in (-Rn^\gamma, Rn^\gamma)$, $C_n^{(2)} \to 2\beta$, and $G_{\beta_n, K_n}^{(4)}(0) \to G_{\beta, K}^{(4)}(0) > 0$.

Given $\gamma \in (0, 1)$, we define

$$v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}. \quad (8.2)$$

In Theorem 6.1 we prove that when $v = 0$, $S_n/n^{1-\gamma}$ satisfies the scaling limit

$$P_{n, \beta_n, K_n}\{S_n/n^{1-\gamma} \in dx\} \Longrightarrow \exp[-G(x)]dx,$$

where

$$G(x) = \delta(v, 2\gamma + \theta - 1)k\beta x^2 + \delta(v, 4\gamma - 1)c_4 x^4$$

and c_4 is defined in (6.5). As enumerated in Table 6.1, the 4 different forms of the limiting density depend on the values of γ and θ and the sign of k.

In Theorem 8.1 we prove the analogous results on the level of MDPs. Assume that the quantity v defined in (8.2) is negative. Then, when (β_n, K_n) is chosen as in Theorem 6.1, $S_n/n^{1-\gamma}$ satisfies the MDP with exponential speed n^{-v} and rate function $\Gamma(x) = G(x) - \inf_{y \in \mathbb{R}} G(y)$, where G is defined in the last display. We prove the MDP in Theorem 8.1 by

showing that when $v < 0$, $S_n/n^{1-\gamma}$ satisfies the Laplace principle with speed n^{-v} and rate function Γ; i.e., for any bounded, continuous function ψ

$$\lim_{n \to \infty} \frac{1}{n^{-v}} \log \int_{\Lambda^n} \exp[n^{-v} \, \psi(S_n/n^{1-\gamma})] \, dP_{n,\beta_n,K_n} = \sup_{x \in \mathbb{R}} \{\psi(x) - \Gamma(x)\}.$$

By Theorem 1.2.3 in [13] the fact that $S_n/n^{1-\gamma}$ satisfies the Laplace principle implies that $S_n/n^{1-\gamma}$ satisfies the LDP with the same speed n^{-v} and the same rate function Γ; i.e., for any closed subset F in $\mathbb{R}$

$$\limsup_{n \to \infty} \frac{1}{n^{-v}} \log P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in F\} \le - \inf_{x \in F} \Gamma(x)$$

and for any open subset Φ in $\mathbb{R}$

$$\liminf_{n \to \infty} \frac{1}{n^{-v}} \log P_{n,\beta_n,K_n}\{S_n/n^{1-\gamma} \in \Phi\} \ge - \inf_{x \in \Phi} \Gamma(x).$$

Γ is obviously a rate function. One easily checks that in all 4 cases given in part (b) of Theorem 8.1 $-v < 1$. Hence $n^{-v}/n \to 0$ as $n \to \infty$, and so we have an MDP. In cases 1, 2, and 3, we have $\inf_{y \in \mathbb{R}} G(y) = 0$ and thus $\Gamma = G$; in case 4, $\inf_{y \in \mathbb{R}} G(y) < 0$.

As in the scaling limits in Theorem 6.1, the rate function in Theorem 8.1 takes the 4 forms enumerated in cases 1, 2, 3, and 4 in Table 8.1. In case 2 the requirement that $G(x) \to \infty$ as $|x| \to \infty$ forces $k > 0$. By contrast, in case 4, $k < 0$ is allowed. In case 1 we can also choose k to be any real number; this affects only the definition of the sequence K_n, not the form of the rate function.

The forms of the rate functions reflect the influence, respectively, of B, of A, and of A and B. In each case the particular set or sets that influence the form of G depend on the speed at which (β_n, K_n) approaches $(\beta, K_c(\beta))$ and the direction of approach. Case 2, which corresponds to the influence of A alone, has two subcases, labeled 2a and 2b in Table 8.1.

In Figure 5 and in Table 8.1 we indicate the subsets of the positive quadrant of the θ-γ plane leading to the 4 cases of the MDPs in Theorem 8.1. Subcases 2a and 2b correspond, respectively, to the left half and the right half of the triangle labeled A in Figure 5. An interesting connection between the MDPs in Theorem 8.1 and the scaling limits in Theorem 6.1 is revealed by comparing Figure 5 with Figure 4, which exhibits the subsets of the positive quadrant of the θ-γ plane leading to the 4 cases of the scaling limits in Theorem 6.1. The subsets labeled A, B, and $A + B$ in Figure 4 are each a subset of the boundary of the set having the same label in Figure 5. The relevant boundaries in Figure 5 are labeled $\partial^+ A$, $\partial^+ B$, and $\partial^+(A + B)$, the first two of which are indicated by dotted lines. This relationship between the two figures is not a surprise because the sets labeled A, B, and $A + B$ in Figure 4 are determined by solving $v = 0$ while the sets having the same labels in Figure 5 are determined by solving $v < 0$.
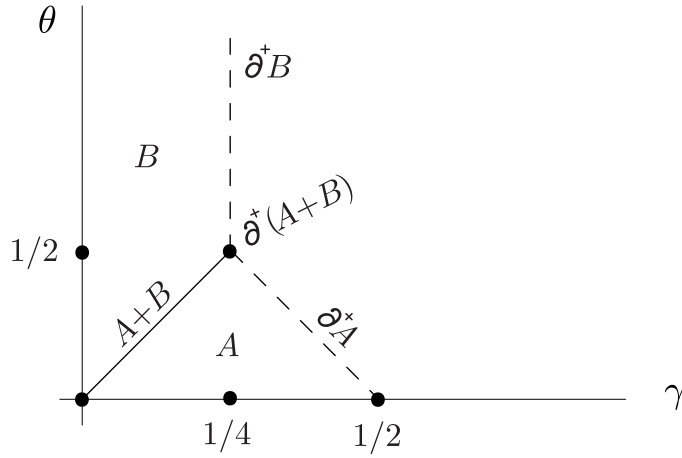
Figure 5: Influence of B and A on MDPs when $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$

Theorem 8.1. *For fixed $\beta \in (0, \beta_c)$, let β_n be an arbitrary positive sequence that converges to β. Given $\theta > 0$ and $k \neq 0$, define*

$$K_n = K(\beta_n) - k/n^\theta,$$

where $K(\beta) = (e^\beta + 2)/(4\beta)$ for $\beta > 0$. Then $(\beta_n, K_n) \to (\beta, K_c(\beta)) \in B$. Given $\gamma \in (0, 1)$, we also define

$$G(x) = \delta(v, 2\gamma + \theta - 1)k\beta x^2 + \delta(v, 4\gamma - 1)c_4 x^4, \tag{8.3}$$

where $c_4 > 0$ is given by

$$c_4 = \frac{G^{(4)}_{\beta, K_c(\beta)}(0)}{4!} = \frac{2[2\beta K_c(\beta)]^4(4 - e^\beta)}{4!(e^\beta + 2)^2} = \frac{(e^\beta + 2)^2(4 - e^\beta)}{2^3 \cdot 4!}.$$

The following conclusions hold.

(a) *Assume that $v = \min\{2\gamma + \theta - 1, 4\gamma - 1\}$ satisfies $v < 0$. Then with respect to P_{n,β_n,K_n}, $S_n/n^{1-\gamma}$ satisfies the Laplace principle, and thus the MDP, with exponential speed n^{-v} and rate function $\Gamma(x) = G(x) - \inf_{y \in \mathbb{R}} G(y)$.*

(b) *We have $v < 0$ if and only if one of the 4 cases enumerated in Table 8.1 holds. Each of the 4 cases corresponds to a set of values of γ and θ, a choice of sign of k, the influence of one or more sets B and A, and a particular exponential speed and a particular form of the rate function in part* (a). *The function G appearing in the definition of the rate function is shown in column 5 in Table 8.1; in case 4 the nonzero constant $\inf_{y \in \mathbb{R}} G(y)$ in the definition of the rate function is not shown. In case 1 the choice of $k \in \mathbb{R}$ does not affect the form of the rate function.*

case influence	values of γ	values of θ	exp'l speed	function G in rate function Γ
1 B	$\gamma \in (0, \frac{1}{4})$	$\theta > 2\gamma$	$n^{1-4\gamma}$	$c_4 x^4$ $c_4 > 0,\ k \in \mathbb{R}$
2a A	$\gamma \in (0, \frac{1}{4}]$	$\theta \in (0, 2\gamma)$	$n^{1-2\gamma-\theta}$	$k\beta x^2$ $k > 0$
2b A	$\gamma \in (\frac{1}{4}, \frac{1}{2})$	$\theta \in (0, 1 - 2\gamma)$	$n^{1-2\gamma-\theta}$	$k\beta x^2$ $k > 0$
3–4 A + B	$\gamma \in (0, \frac{1}{4})$	$\theta = 2\gamma$	$n^{1-4\gamma}$	$k\beta x^2 + c_4 x^4$ $k \neq 0$

Table 8.1*: Values of γ and θ, exponential speeds, and rate functions in part (b) of Theorem 8.1*

Proof. We first prove part (b) from part (a) and then prove part (a).

(b) We have $v < 0$ in the following 4 mutually exclusive and exhaustive cases. As (8.3) makes clear, $v = 4\gamma - 1 < 0$ corresponds to the influence of B and $v = 2\gamma + \theta - 1$ to the influence of A.

- **Case 1: Influence of B alone.** $v = 4\gamma - 1 < 0$, $4\gamma - 1 < 2\gamma + \theta - 1$, and $k \in \mathbb{R}$. In this case $\gamma \in (0, 1/4)$ and $\theta > 2\gamma$, which corresponds to the second and third columns for case 1 in Table 8.1.

- **Case 2: Influence of A alone.** $v = 2\gamma + \theta - 1 < 0$, $2\gamma + \theta - 1 < 4\gamma - 1$, and $k > 0$. In this case $0 < \theta < \min\{2\gamma, 1 - 2\gamma\}$. Since $0 < 2\gamma \leq 1 - 2\gamma \Leftrightarrow \gamma \in (0, 1/4]$ and $0 < 1 - 2\gamma < 2\gamma \Leftrightarrow \gamma \in (1/4, 1/2)$, case 2 corresponds to the second and third columns for case 2a and case 2b in Table 8.1.

- **Cases 3–4: Influence of A and B.** $v = 4\gamma - 1 = 2\gamma + \theta - 1 < 0$, $k > 0$ for case 3, and $k < 0$ for case 4. In these cases $0 < \gamma < 1/4$ and $\theta = 2\gamma$. Hence case 3–4 correspond to the second and third columns for cases 3–4 in Table 8.1.

In cases 1, 2, 3, and 4 we have, respectively, $G(x) = c_4 x^4$, $G(x) = k\beta x^2$ with $k > 0$, $G(x) = k\beta x^2 + c_4 x^4$ with $k > 0$, and $G(x) = k\beta x^2 + c_4 x^4$ with $k < 0$. In combination with part (a), we obtain the 4 rate functions given in the last column of Table 8.1.

(a) Our strategy is to prove that with respect to $P_{n,\beta_n,K_n} \times Q$, $S_n/n^{1-\gamma} + W_n/n^{1/2-\gamma}$ satisfies the Laplace principle with exponential speed n^{-v} and rate function Γ. In order to prove the Laplace principle for $S_n/n^{1-\gamma}$ alone, we need the following estimate, which shows that

$W_n/n^{1/2-\gamma}$ is superexponentially small relative to $\exp(n^{-\upsilon})$: for any $\delta > 0$

$$\limsup_{n\to\infty} \frac{1}{n^{-\upsilon}} \log Q\{|W_n/n^{1/2-\gamma}| > \delta\} = -\infty. \tag{8.4}$$

According to Theorem 1.3.3 in [13], if with respect to $P_{n,\beta_n,K_n} \times Q$, $W_n/n^{1/2-\gamma} + S_n/n^{1-\gamma}$ satisfies the Laplace principle with speed $n^{-\upsilon}$ and rate function Γ, then with respect to P_{n,β_n,K_n}, $S_n/n^{1-\gamma}$ satisfies the Laplace principle with speed $n^{-\upsilon}$ and rate function Γ. Since the Laplace principle implies the MDP [13, Thm. 1.2.3], part (a) of the present theorem will be proved.

We now prove (8.4). Denote the variance $(2\beta_n K_n)^{-1}$ of W_n by σ_n^2. Since β_n and K_n are bounded and uniformly positive over n, the sequence σ_n^2 is bounded and uniformly positive over n. We have the inequality

$$\begin{aligned}
Q\{|W_n/n^{1/2-\gamma}| > \delta\} &= Q\{|N(0,\sigma_n^2)| > n^{1/2-\gamma}\delta\} \\
&\leq \frac{\sqrt{2}\sigma_n}{\sqrt{\pi}n^{1/2-\gamma}\delta} \cdot \exp(-n^{1-2\gamma}\delta^2/[2\sigma_n^2]).
\end{aligned}$$

Hence (8.4) follows if $1 - 2\gamma > -\upsilon$. Since γ and θ are both positive, this is easily verified to hold when either $\upsilon = 4\gamma - 1$ or $\upsilon = 2\gamma + \theta - 1$.

We now turn to the Laplace principle for $S_n/n^{1-\gamma} + W_n/n^{1/2-\gamma}$. Let ψ be an arbitrary bounded, continuous function. Choosing $f = \exp[n^{-\upsilon}\psi]$ in Lemma 4.1 yields

$$\int_{\Lambda^n \times \Omega} \exp\left[n^{-\upsilon}\,\psi\left(\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}}\right)\right] d(P_{n,\beta_n,K_n} \times Q) \tag{8.5}$$

$$= \frac{1}{\int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)]\,dx} \cdot \int_{\mathbb{R}} \exp[n^{-\upsilon}\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]\,dx,$$

In order to obtain the appropriate expansion of $nG_{\beta_n,K_n}(x/n^\gamma)$ in this display, we multiply the numerator and denominator of the right hand side of (8.1) by $n^{-\upsilon}$, obtaining

$$nG_{\beta_n,K_n}(x/n^\gamma) = n^{-\upsilon}G_n(x),$$

where

$$G_n(x) = \frac{1}{n^{2\gamma+\theta-1-\upsilon}}\frac{C_n^{(2)}}{2!}x^2 + \frac{1}{n^{4\gamma-1-\upsilon}}\frac{G_{\beta_n,K_n}^{(4)}(0)}{4!}x^4 + \frac{1}{n^{5\gamma-1-\upsilon}}B_n(\xi(x/n^\gamma))x^5.$$

The proof of the Laplace principle for $S_n/n^{1-\gamma} + W_n/n^{1/2-\gamma}$ rests on the following properties of $nG_{\beta_n,K_n}(x/n^\gamma) = n^{-\upsilon}G_n(x)$, which in turn are consequences of the Taylor expansion of $G_n(x)$ just given. Because of the estimate (8.4) on $W_n/n^{1/2-\gamma}$, the inequality in (8.6), and

the uniform convergence of G_n to G expressed in item 3 below, the proof of the MDPs, though analogous, is more delicate than the proof of the scaling limits in section 6, for which the a.s. convergence of $W_n/n^{1/2-\gamma}$ to 0, the pointwise convergence of $G_{\beta_n, K_n}(x/n^\gamma)$ to $G(x)$, and the lower bound (6.7) suffice.

1. There exists $R > 0$ and a polynomial H with the properties that $H(x) \to \infty$ as $|x| \to \infty$ and for all sufficiently large n and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$nG_{\beta_n, K_n}(x/n^\gamma) \geq n^{-\upsilon} H(x).$$

   In case 1 when $k \geq 0$ as well as in cases 2 and 3, $H(x) = G(x)/2$; in case 1 when $k < 0$ and in case 4, which corresponds to $k < 0$, $H(x) = -2|k|\beta x^2 + c_4 x^4/2$.

2. Let $\Delta = \sup_{x \in \mathbb{R}} \{\psi(x) - G(x)\}$. Since $H(x) \to \infty$ and $G(x) \to \infty$, there exists $M > 0$ with the properties that

$$\sup_{|x|>M} \{\psi(x) - H(x)\} \leq -|\Delta| - 1,$$

   the supremum of $\psi - G$ on $\mathbb{R}$ is attained on the interval $[-M, M]$, and the supremum of $-G$ on $\mathbb{R}$ is attained on the interval $[-M, M]$. In combination with item 1, we see that for all $n \in \mathbb{N}$ satisfying $Rn^\gamma > M$

$$\sup_{M<|x|<Rn^\gamma} \{n^{-\upsilon}\psi(x) - nG_{\beta_n, K_n}(x/n^\gamma)\} \leq -n^{-\upsilon}(|\Delta| + 1). \tag{8.6}$$

3. Let M be the number selected in item 2. Then for all $x \in \mathbb{R}$ satisfying $|x| \leq M$, $G_n(x) = n^{1+\upsilon} G_{\beta_n, K_n}(x/n^\gamma)$ converges uniformly to $G(x)$ as $n \to \infty$.

Since $nG_{\beta_n, K_n}(x/n^\gamma) = n^{-\upsilon} G_n(x)$, item 3 implies that for any $\delta > 0$ and all sufficiently large n

$$\exp(-n^{-\upsilon}\delta) \int_{\{|x|\leq M\}} \exp[n^{-\upsilon}(\psi(x) - G(x))] \, dx$$
$$\leq \int_{\{|x|\leq M\}} \exp[n^{-\upsilon}\psi(x) - nG_{\beta_n, K_n}(x/n^\gamma)] \, dx$$
$$\leq \exp(n^{-\upsilon}\delta) \int_{\{|x|\leq M\}} \exp[n^{-\upsilon}(\psi(x) - G(x))] \, dx.$$

In addition, item 2 implies that

$$\int_{\{M<|x|<Rn^\gamma\}} \exp[n^{-\upsilon}\psi(x) - nG_{\beta_n, K_n}(x/n^\gamma)] \, dx \leq 2Rn^\gamma \exp[-n^{-\upsilon}(|\Delta| + 1)].$$

Since ψ is bounded, the last two displays show that there exist $a_5 > 0$ and $a_6 \in \mathbb{R}$ such that for all sufficiently large n

$$\int_{\{|x|<Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)] \, dx \le a_5 \exp(n^{-v}a_6).$$

Since $-v \in (0,1)$, we conclude from part (d) of Lemma 4.4 the existence of $a_7 > 0$ such that for all sufficiently large n

$$\int_{\{|x|\ge Rn^\gamma\}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)] \, dx \le 2a_5 \exp(-na_7).$$

We now put these three estimates together. For all sufficiently large n we have

$$\exp(-n^{-v}\delta) \int_{\{|x|\le M\}} \exp[n^{-v}(\psi(x) - G(x))] \, dx$$

$$\le \int_\mathbb{R} \exp[n^{-v}\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)] \, dx$$

$$\le \exp(n^{-v}\delta) \int_{\{|x|\le M\}} \exp[n^{-v}(\psi(x) - G(x))] \, dx + \delta_n,$$

where

$$\delta_n \le 2Rn^\gamma \exp[-n^{-v}(|\Delta| + 1)] + 2a_5 \exp(-na_7 + n^{-v}\|\psi\|_\infty).$$

Since $-v < 1$ and since by item 2

$$\lim_{n\to\infty} \frac{1}{n^{-v}} \log \int_{\{|x|\le M\}} \exp[n^{-v}(\psi(x) - G(x))] \, dx$$
$$= \sup_{|x|\le M} \{\psi(x) - G(x)\} = \sup_{x\in\mathbb{R}}\{\psi(x) - G(x)\},$$

we have

$$\sup_{x\in\mathbb{R}}\{\psi(x) - G(x)\} - \delta$$

$$\le \liminf_{n\to\infty} \frac{1}{n^{-v}} \log \int_\mathbb{R} \exp[n^{-v}\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)] \, dx$$

$$\le \limsup_{n\to\infty} \frac{1}{n^{-v}} \log \int_\mathbb{R} \exp[n^{-v}\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)] \, dx$$
$$\le \sup_{x\in\mathbb{R}}\{\psi(x) - G(x)\} + \delta,$$

and because $\delta > 0$ is arbitrary, it follows that

$$\lim_{n\to\infty} \frac{1}{n^{-v}} \log \int_{\mathbb{R}} \exp[n^{-v}\psi(x) - nG_{\beta_n,K_n}(x/n^\gamma)]\, dx = \sup_{x\in\mathbb{R}}\{\psi(x) - G(x)\}.$$

Combining this limit with the same limit for $\psi = 0$, we conclude from (8.5) that

$$\lim_{n\to\infty} \frac{1}{n^{-v}} \log \int_{\Lambda^n\times\Omega} \exp\left[n^{-v}\psi\left(\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}}\right)\right] d(P_{n,\beta_n,K_n}\times Q)$$
$$= \sup_{x\in\mathbb{R}}\{\psi(x) - G(x)\} + \inf_{y\in\mathbb{R}} G(y) = \sup_{x\in\mathbb{R}}\{\psi(x) - \Gamma(x)\}.$$

This completes the proof that with respect to $P_{n,\beta_n,K_n}\times Q$, $S_n/n^{1-\gamma} + W_n/n^{1/2-\gamma}$ satisfies the Laplace principle with exponential speed n^{-v} and rate function Γ. Since $W_n/n^{1/2-\gamma}$ is superexponentially small, we obtain the desired Laplace principle for $S_n/n^{1-\gamma}$ with respect to P_{n,β_n,K_n}. The proof of the theorem is complete. ∎

We next formulate the MDP for $S_n/n^{1-\gamma}$ when (β_n, K_n) is an arbitrary positive sequence that converges to $(\beta, K) \in A$; thus β and K satisfy $0 < \beta \le \beta_c$ and $0 < K < K_c(\beta)$. Because in this case the normal random variable W_n contributes to the limit, we are not able to prove the MDP as we proved Theorem 8.1. Instead we use the Gärtner-Ellis Theorem. The following theorem is also valid for $\beta > \beta_c$ and $0 < K < K_c(\beta)$, and the proof is essentially the same. The key observation is that for $\beta > \beta_c$, we have $K(\beta) = (e^\beta + 2)/(4\beta) > K_c(\beta)$ [22, Thm. 3.8]. Hence if $K < K_c(\beta)$, then also $K < K(\beta)$ and thus $G^{(2)}_{\beta,K}(0)$ in (8.7) is positive.

Theorem 8.2. *Let (β_n, K_n) be an arbitrary positive sequence that converges to $(\beta, K) \in A$. Let γ be any number in $(0, 1/2)$. Then with respect to P_{n,β_n,K_n}, $S_n/n^{1-\gamma}$ satisfies the MDP with exponential speed $n^{1-2\gamma}$ and rate function $\beta[K(\beta) - K]x^2$. Thus the limit is independent of the particular sequence (β_n, K_n) that is chosen.*

Proof. For $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we use the monotone convergence theorem to replace f in Lemma 4.1 by $\exp(n^{1-2\gamma}tx)$. We then use the Taylor expansion in part (a) of Theorem 4.3 and the fact that $G^{(2)}_{\beta_n,K_n}(0)$ given in (4.9) converges to

$$G^{(2)}_{\beta,K}(0) = \frac{2\beta K[K(\beta) - K]}{K(\beta)}, \tag{8.7}$$

which is positive since $0 < K < K_c(\beta) = K(\beta)$. As in the proof of part (a) of Theorem 8.1, there exists $M > 0$ such that the supremum of $tx - G^{(2)}_{\beta,K}(0)x^2/2$ is attained on the interval

$[-M, M]$ and the following calculation is valid:

$$\lim_{n\to\infty} \frac{1}{n^{1-2\gamma}} \log \int_{\Lambda^n \times \Omega} \exp\left[n^{1-2\gamma}t\left(\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}}\right)\right] d(P_{n,\beta_n,K_n} \times Q)$$

$$= \lim_{n\to\infty} \frac{1}{n^{1-2\gamma}} \log \int_{\mathbb{R}} \exp\left[n^{1-2\gamma}tx - nG_{\beta_n,K_n}(x/n^\gamma)\right] dx$$

$$- \lim_{n\to\infty} \frac{1}{n^{1-2\gamma}} \log \int_{\mathbb{R}} \exp[-nG_{\beta_n,K_n}(x/n^\gamma)] dx$$

$$= \lim_{n\to\infty} \frac{1}{n^{1-2\gamma}} \log \int_{\{|x|\le M\}} \exp\left[n^{1-2\gamma}\left(tx - G^{(2)}_{\beta,K}(0)x^2/2\right)\right] dx$$

$$- \lim_{n\to\infty} \frac{1}{n^{1-2\gamma}} \log \int_{\{|x|\le M\}} \exp\left[-G^{(2)}_{\beta,K}(0)x^2/2\right] dx$$

$$= \sup_{\{|x|\le M\}} \left\{tx - G^{(2)}_{\beta,K}(0)x^2/2\right\} + \inf_{\{|x|\le M\}} \left\{G^{(2)}_{\beta,K}(0)x^2/2\right\}$$

$$= \frac{t^2}{2G^{(2)}_{\beta,K}(0)}.$$

Since W_n is an $N(0, (2\beta_n K_n)^{-1})$ random variable and is independent of S_n,

$$\lim_{n\to\infty} \frac{1}{n^{1-2\gamma}} \log \int_{\Lambda^n} \exp\left[n^{1-2\gamma}t \cdot \frac{S_n}{n^{1-\gamma}}\right] dP_{n,\beta_n,K_n}$$

$$= \lim_{n\to\infty} \frac{1}{n^{1-2\gamma}} \log \int_{\Lambda^n \times \Omega} \exp\left[n^{1-2\gamma}t\left(\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}}\right)\right] d(P_{n,\beta_n,K_n} \times Q)$$

$$- \lim_{n\to\infty} \frac{1}{n^{1-2\gamma}} \log \int_{\Omega} \exp\left[n^{1/2-\gamma}tW_n\right] dQ$$

$$= \frac{t^2}{2G^{(2)}_{\beta,K}(0)} - \frac{t^2}{4\beta K} = \frac{t^2}{2} \cdot \frac{1}{2\beta[K(\beta) - K]}.$$

The Gärtner-Ellis Theorem [16] now implies that $S_n/n^{1-\gamma}$ satisfies the MDP with exponential speed $n^{1-2\gamma}$ and rate function

$$I(x) = \sup_{t\in\mathbb{R}}\left\{tx - \frac{t^2}{2} \cdot \frac{1}{2\beta[K(\beta) - K]}\right\} = \beta[K(\beta) - K]x^2.$$

This completes the proof. ■

In the context of the proof of the preceding theorem, it is worthwhile pointing out that the Gärtner-Ellis Theorem cannot be used to prove all the other MDPs for $S_n/n^{1-\gamma}$ in this section.

For example, consider the MDPs in Theorem 8.1. For any $t \in \mathbb{R}$ one calculates

$$
\begin{aligned}
g(t) &= \lim_{n \to \infty} \frac{1}{n^{-v}} \log \int_{\Lambda^n \times \Omega} \exp\left[n^{-v} t\left(\frac{S_n}{n^{1-\gamma}} + \frac{W_n}{n^{1/2-\gamma}}\right)\right] d(P_{n,\beta_n,K_n} \times Q) \\
&= \sup_{x \in \mathbb{R}}\{tx - G(x)\} + \inf_{y \in \mathbb{R}} G(y) = \sup_{x \in \mathbb{R}}\{tx - [G(x) - \bar{G}]\},
\end{aligned}
$$

where $\bar{G} = \inf_{y \in \mathbb{R}} G(y)$. Thus g equals the Legendre-Fenchel transform of $G - \bar{G}$. If $G - \bar{G}$ is strictly convex on $\mathbb{R}$, as it is in cases 1, 2, and 3 in Theorem 8.1, then g is differentiable on $\mathbb{R}$ [32, p. 253]. Hence by the Gärtner-Ellis Theorem, $S_n/n^{1-\gamma}$ satisfies the MDP with exponential speed n^{-v} and rate function given by the Legendre-Fenchel transform of g, which is $G - \bar{G}$. In cases 1, 2, and 3 in Theorem 8.1, $\bar{G}$ equals 0, and we recover the form of the rate function in column 4 of Table 8.1. However, the situation is different in the MDP in case 4, in which $G(x) = k\beta x^2 + c_4 x^4$ with $k < 0$. Here $\bar{G} < 0$, G is not convex on all of $\mathbb{R}$, and g is not differentiable on $\mathbb{R}$. As a result, the Gärtner-Ellis Theorem cannot be applied to obtain the lower large deviation bound for all open sets and thus to obtain the MDP. In addition, the Legendre-Fenchel transform of g equals 0 on a symmetric interval containing the origin, and thus it does not coincide with $G - \bar{G}$ on this interval. A similar situation holds in Theorem 8.3, in which we derive 13 MDPs for suitable sequences $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$. In cases 1–4, 6, 8, and 10 in that theorem, the coefficients in the polynomial G are all positive, and so G is strictly convex and $\bar{G} = 0$. Hence the corresponding MDPs can be derived via the Gärtner-Ellis Theorem. However, in all the other cases except for case 12 with k sufficiently large, the polynomial G is not convex on all of $\mathbb{R}$; as in case 4 in Theorem 8.1, the Gärtner-Ellis Theorem cannot be applied to obtain the MDP.

We now consider the final class of MDPs in this section. This class arises when (β_n, K_n) converges to $(\beta_c, K_c(\beta_c))$ along the same sequences considered in Theorem 7.1, where we proved scaling limits for $S_n/n^{1-\gamma}$ for $\gamma \in (0, 1/2)$. Given $\alpha > 0$, $\theta > 0$, $b \neq 0$, and $k \neq 0$, these sequences are defined by

$$
\beta_n = \log(4 - b/n^\alpha) = \log(e^{\beta_c} - b/n^\alpha) \text{ and } K_n = K(\beta_n) - k/n^\theta. \tag{8.8}
$$

For these sequences the parameter that plays the role of v in Theorem 8.1 is

$$
w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}.
$$

The 13 forms of the scaling limits of $S_n/n^{1-\gamma}$ are proved in Theorem 7.1 under the assumption that $w = 0$. We now assume that $w < 0$. Using the same Taylor expansion that was used to deduce these scaling limits [Thm. 4.3(c)], one deduces the 13 forms of the Laplace principles for $S_n/n^{1-\gamma}$. These Laplace principles and the equivalent MDPs are stated in the next theorem along with the choices of γ, α, b, θ, and k leading to the 13 forms of the rate function. The only

requirement on b and k is that $G(x) \to \infty$ as $|x| \to \infty$. This requirement forces $b > 0$ in case 2 and $k > 0$ in case 3. The proof of the MDPs in the next theorem is omitted because it follows the same pattern of proof of Theorem 8.1.

As in Theorem 7.1, there are further possibilities concerning the sign of b and k. In all the cases in which no x^4 term appears in the scaling limit (cases 1, 3, 6, 7), we can choose either b to be any real number. Similarly, in all the cases in which no x^2 term appears in the scaling limit (cases 1, 2, 4, 5), we can choose either k to be any real number. Although the choice of b or k affects the definition of the sequence (β_n, K_n), it does not affect the form of the rate function.

Theorem 8.3. *Given $\alpha > 0$, $\theta > 0$, $b \neq 0$, and $k \neq 0$, consider the sequence (β_n, K_n) defined in (8.8). Then $(\beta_n, K_n) \to (\beta_c, K_c(\beta_c))$. Given $\gamma \in (0, 1)$, we also define*

$$G(x) = \delta(w, 2\gamma + \theta - 1)k\beta_c x^2 + \delta(w, 4\gamma + \alpha - 1)b\bar{c}_4 x^4 + \delta(w, 6\gamma - 1)c_6 x^6,$$

where $\bar{c}_4 = 3/16$ and $c_6 = 9/40$. The following conclusions hold.

(a) Assume that $w = \min\{2\gamma + \theta - 1, 4\gamma + \alpha - 1, 6\gamma - 1\}$ satisfies $w < 0$. Then with respect to P_{n,β_n,K_n}, $S_n/n^{1-\gamma}$ satisfies the Laplace principle, and thus the MDP, with exponential speed n^{-w} and rate function $\Gamma(x) = G(x) - \inf_{y \in \mathbb{R}} G(y)$.

(b) We have $w < 0$ if and only if one of the 13 cases enumerated in Table 8.2 holds. Each of the 13 cases corresponds to a set of values of γ, α, and θ; a choice of signs of b and k; the influence of one or more sets C, B, A; and a particular exponential speed and a particular form of the rate function in part (a). The function G appearing in the definition of the rate function is shown in column 5 in Table 8.2; when $\inf_{y \in \mathbb{R}} G(y) \neq 0$, this additive constant in the definition of the rate function is not shown. The form of the rate function is not affected by the choice of $b \in \mathbb{R}$ in cases 1, 3, 6, and 7 and by the choice of $k \in \mathbb{R}$ in cases 1, 2, 4, and 5.

case influence	values of γ	values of α values of θ	exp'l speed	function G in rate function Γ
1 C	$\gamma \in (0, \frac{1}{6})$	$\alpha > 2\gamma$ $\theta > 4\gamma$	$n^{1-6\gamma}$	$c_6 x^6$ $c_6 > 0,\ b \in \mathbb{R},\ k \in \mathbb{R}$
2 B	$\gamma \in (0, \frac{1}{4})$	$\alpha \in (0, \min\{2\gamma, 1-4\gamma\})$ $\theta > 2\gamma + \alpha$	$n^{1-4\gamma-\alpha}$	$b\bar{c}_4 x^4$ $b > 0,\ \bar{c}_4 > 0,\ k \in \mathbb{R}$
3 A	$\gamma \in (0, \frac{1}{2})$	$\theta \in (0, \min\{4\gamma, 1-2\gamma\})$ $\alpha > \max(\theta - 2\gamma, 0)$	$n^{1-2\gamma-\theta}$	$k\beta_c x^2$ $k > 0,\ b \in \mathbb{R}$
4–5 $B+C$	$\gamma \in (0, \frac{1}{6})$	$\alpha = 2\gamma$ $\theta > 4\gamma$	$n^{1-6\gamma}$	$b\bar{c}_4 x^4 + c_6 x^6$ $b \neq 0,\ k \in \mathbb{R}$
6–7 $A+C$	$\gamma \in (0, \frac{1}{6})$	$\alpha > 2\gamma$ $\theta = 4\gamma$	$n^{1-6\gamma}$	$k\beta_c x^2 + c_6 x^6$ $k \neq 0,\ b \in \mathbb{R}$
8–9 $A+B$	$\gamma = (0, \frac{1}{4})$	$\alpha \in (0, \min\{2\gamma, 1-4\gamma\})$ $\theta = 2\gamma + \alpha$	$n^{1-4\gamma-\alpha}$	$k\beta_c x^2 + b\bar{c}_4 x^4$ $k \neq 0,\ b > 0$
10–13 $A+B+C$	$\gamma \in (0, \frac{1}{6})$	$\alpha = 2\gamma$ $\theta = 4\gamma$	$n^{1-6\gamma}$	$k\beta_c x^2 + b\bar{c}_4 x^4 + c_6 x^6$ $k \neq 0,\ b \neq 0$

Table 8.2: Values of γ, α, and θ, exponential speeds, and rate functions in part (b) of Theorem 8.3

As discussed in section 2, the MDPs listed in Table 8.2 yield a new class of distribution limits for $S_n/n^{1-\gamma}$ in those cases in which the set of global minimum points of G contains nonzero points. These are the cases in which the coefficients of G are not all positive: cases 5 ($b < 0$), 7 ($k < 0$), 9 ($k < 0$), 11 ($k < 0,\ b > 0$), 12 ($k > 0,\ b < 0$), and 13 ($k < 0$, $b < 0$). In all these cases except for case 12, we obtain the limit (2.13). Case 12 exhibits the most complicated behavior, giving rise to the limit (2.14) for the critical value $k = 5b^2/[2^7\beta_c]$. These limits and the underlying physical phenomena are now being investigated for a class of non-mean-field models, including the Blume-Emery-Griffiths model [19].

This completes our study of limit theorems for the BEG model in the neighborhood of the tricritical point $(\beta_c, K_c(\beta_c)) \in C$, in the neighborhood of second-order points $(\beta, K_c(\beta)) \in B$, and in the neighborhood of single-phase points $(\beta, K) \in A$. It is an unexpectedly rich and fruitful area of research, one that we hope will inspire similar investigations for other statistical mechanical models.

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